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An unstable uniform slab model of the mixed layer as a source of downward propagating near-inertial motion.

Part 1: Steady mean flow

by John Kroll¹

ABSTRACT

The linear stability of a one layer turbulent slab model of the mixed layer over a continuously stratified, inviscid, infinite depth ocean is investigated for large horizontal scale \((\geq 0 \text{ (1 km)})\) perturbations. We assume the steady state values for the velocity, density and depth of the mixed layer are essentially uniform in time and space and use eddy viscosity in the perturbation equations. The model represents an extreme case in that it suppresses the analog of the so called inviscid, inflection point instability of the laminar problem and allows only the analog of the so called viscous, parallel flow instability. An analysis of two differing unstable flow scenarios indicates a critical point of instability at a near-inertial frequency in many cases and at a horizontal wavelength consistent with observations beneath the mixed layer (0 (10 km)). This implies that the Ekman transport could be a source of downward propagating inertial motion. No horizontal variation in the wind to produce Ekman suction in the usual manner would be necessary.

1. Introduction

In the ocean it has been observed that internal wave energy near the local inertial frequency, \(f\), propagates downward from the surface (Leamon and Sanford (1975), Sanford (1975), Kundu (1976), Rossby and Sanford (1976), Pollard (1980)). The most apparent mechanism to accomplish this is a horizontally varying wind producing Ekman suction near the surface with a frequency slightly greater than inertial. The suction in turn will generate downward propagating near-inertial motion into the deeper ocean (see Kroll, 1975).

However, the scale of the inertial motion beneath the mixed layer seems to have no relation to any dominant scale in the wind (e.g., see Sanford or Pollard). A favored scale unrelated to forcing scale suggests the type of behavior of an unstable flow. Therefore, our aim will be to examine the linear stability of the simplest turbulent mixed layer flow possible, the uniform slab, looking for critical values of the frequency and wave number consistent with general observations.

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The investigation by Stern (1977) of over-reflection of inertial waves by the mixed layer is related to a stability analysis. His analysis would imply that waves reflecting back and forth between the surface and bottom can be amplified. It would seem that the inertial energy beneath the mixed layer in a shallow sea might be explained by this, but it would seem less likely that this effect would take place in a deep ocean. This would be because a wave packet produced at a given location at the surface will travel the order of 100 km horizontally and take weeks to travel to the bottom and return in an order 5 km deep ocean. Over this time and distance one would expect the general variability of the ocean to substantially change its character if not totally decimate it. Our model will be further contrasted with the Stern model and other models in what follows.

In Sections 2 and 3 we describe the model and derive the perturbation equations. In Section 4 we discuss the perturbation energy flux balance in the mixed layer. In Section 5 we outline the stability analysis of the perturbation equations. In Section 6 we discuss the solution for the special cases: (a) vanishing horizontal eddy viscosity, (b) constant Brunt-Väisälä frequency in the thermocline and (c) no density jump across the interface between the mixed layer and thermocline. In Section 7 we discuss the solution in general, and in Section 8 we describe and compare two unstable flow scenarios. We state our conclusions in Section 9.

2. The model

The steady state flow for the mixed layer is the uniform slab model of Pollard et al. (1973) without inertial oscillations. Hence the steady flow is described by \( fU = \frac{1}{\rho_1} \frac{\partial \tau^y(z)}{\partial z} \), where \( U \) is the steady flow in the \( x \) direction, \( f \) the Coriolis parameter, \( \rho_1 \) the density, and \( \tau^y(z) \) the mean turbulent Reynolds stress in the \( y \) direction (\( \tau^y(0) \) is the wind stress at the surface). The density is assumed constant and the stress is assumed to vary linearly with \( z \), so that the \( U \) is constant and positive.

The perturbation equations must be derived with care taking into account the limiting process assumed in obtaining the steady state. We assume a transition region of thickness \( \epsilon \to 0 \) between the uniform mixed layer described above and the quiescent fluid beneath. If in the perturbation system, we assume no perturbation of the Reynolds stress we will obtain an unbounded perturbation of the horizontal velocity and a jump in the vertical velocity in the transition region as \( \epsilon \to 0 \). This is what Stern obtains. However, we assume instead that the perturbation of the Reynolds stress in this very chaotic entrainment zone is sufficient to at least make the perturbation of the horizontal velocity finite so that there is no jump in the vertical velocity as \( \epsilon \to 0 \).

For the stability of the Ekman spiral in laminar flow, Lilly (1966) described two types of instability mechanisms: the inviscid, inflection point instability and the
viscous, parallel flow instability. Thus the model we are considering is an extreme case where only an analogous parallel flow type of instability is possible as opposed to the other extreme, the inflection point, which Stern considers. This will explain why he has the most unstable wave traveling perpendicular to the flow while ours will be parallel. For the steady state structure considered here, our extreme seems more realistic. However, for a more realistic steady state, we might expect the most unstable wave somewhere between the extremes.

Taking into account the assumptions discussed above, the model to be considered is shown schematically in Figure 1. We assume a Boussinesq fluid on an $f$-plane with a rigid top and no heat transfer at the top. In the turbulent mixed layer we assume that the mean steady state density, $\rho_1$, and velocity, $U$, are spatially and temporally uniform. The depth of the mixed layer, $h$, is assumed constant and the perturbed interface displacement, $\eta$, is assumed $<< h$. The bottom layer is assumed initially quiescent, inviscid and bottomless with a variable steady state density, $\rho_2(z)$. Across the interface there can be a density jump $\Delta \rho = \rho_2(-h) - \rho_1 \geq 0$. The horizontal scale of perturbed motion, $L$, is assumed much greater than $h$ and the scales of all turbulent eddies.

Geometrically, the model resembles the Kelvin-Helmholtz ($K$-$H$) stability problem or closer yet the model considered by Lindzen and Rosenthal (1976). But it is actually quite different from either. The fundamental premise of the model is the assumption that, if an instability exists, the wavelength of the most unstable wave is much greater than $h$. For classical $K$-$H$ flow we know this wavelength approaches zero, short wavelength catastrophe. Hence, for our premise to be possible, the shorter wavelengths must decay more than the longer wavelengths within the scales of wavelengths we are considering ($\geq 0(1 \text{ km})$). This should in fact be true and to mathematically describe it we postulate a horizontal eddy coefficient, $\nu_h$. We will also use a vertical eddy coefficient, $\nu_v$, to approximate the perturbation of the vertical shear stress which will turn out not to enter into any solutions which are
unstable. Using unknown eddy viscosities which are properties of the flow rather than the fluid is never desirable. But, as is often the case, there are no other mathematically tractable alternatives.

The stability of classical K-H flow, as well as the modified versions of Lindzen (1974) and Lindzen and Rosenthal (1976), depends on the stabilizing influence of buoyancy versus the destabilizing influence of inertia represented by a Richardson number. For our model we will find that stability basically depends on stabilizing Coriolis force versus destabilizing inertia represented by a Rossby number. The obvious difference is of course our inclusion of rotation and the assumption that $h/L << 1$ which allows the hydrostatic approximation. However, it should be noted that we show that, even with the inclusion of the nonhydrostatic terms, the instability we obtain is not the continuation of the basic Kelvin-Helmholtz instability with rotation and stratification increasing from zero. Jones (1967) investigated the propagation of internal waves in a shear flow with rotation for a constant Brunt-Väisälä frequency throughout. He showed that the rotation could be neglected in a system with linear shear anywhere that the relation $|\omega + kU| >> f(1 + l^2/k^2)^{1/2}$ was satisfied, where $\omega$ is frequency and $k$ and $l$ are the $x$ and $y$ wave numbers respectively. Our scaling will be such that this relation cannot be assumed anywhere.

3. The perturbation equations

We scale vertical distance $(z)$ by $h$, horizontal distance $(x)$ by $L$ (unspecified for the moment), time by $f_0^{-1}$, horizontal velocity $(v)$, by $U$, vertical velocity $(w)$ by $h/L$, pressure $(p)$ by $\rho_i f_0 UL$, density variation $(\rho)$ by $\rho_i f_0 UL/\rho g$, and interface displacement $(\eta)$ by $hU/Lf_0$. The dimensionless perturbation equations become for the top mixed layer:

$$v_t + Rv_x + f k x v = -p_x i + E v_{zz} + \sigma v_{xx},$$
$$0 = -p_z - \rho,$$
$$\rho_t + R \rho_x = E_H \rho_{xx} + \sigma_H \rho_{xx},$$
$$u_x + w_z = 0,$$

and for the bottom layer:

$$v_t + f k x v = - (p_x + \alpha \eta_z) i,$$
$$0 = -p_z - \rho,$$
$$\rho_t - B^2(z) w = 0,$$
$$u_x + w_z = 0,$$

where $i$ and $k$ are unit vectors in $x$ and $z$ respectively, $E = v_v/f_0 h^2$ (vertical Ekman number), $\sigma = v_h/f_0 L^2$ (horizontal Ekman number), $E_H$ and $\sigma_H$ are the heat transfer equivalents to $E$ and $\sigma$, $R = U/f_0 L$ (Rossby number), $B^2 = (\delta N/f_0)^2$ (Burger num-
ber), \( \alpha = \delta g'/L_{f_0}^2 \) with \( \delta = h/L, g' = g\Delta \rho/\rho_1 \) and \( N = \sqrt{\frac{-g}{\rho_1}} \frac{d\rho_z(z)}{dz} \) (Brunt-Väisälä frequency).

On the \( f \)-plane dimensionless \( f \) is actually unity, but it will be carried through the calculations. We are assuming \( \delta << 1 \) and have neglected terms of \( O(\delta) \) and smaller (but \( \alpha \) and \( B = 0(1) \)). Also, since there is no preferred direction on the \( f \)-plane, Squire's theorem holds and we need assume perturbations in the direction of \( U \) only, the \( x \) direction.

The boundary conditions for the perturbed system are that \( u_z, v_z, \rho_z, \) and \( w \) vanish at \( z = 0 \) and that \( u_z, v_z \) and \( \rho_z \) vanish and \( p \) and \( \eta \) be continuous at \( z = -1 \). Also, there must be no radiation of energy form \( z \to -\infty \). In fact we assume that the bottom layer has no perturbation sources which means we can assume downward energy flux throughout the bottom layer. The kinematic condition relating \( \eta \) and \( w \) is \( \frac{d\eta}{dt} = w \) which in dimensionless form becomes \( \eta_t + R\eta_x = w \) just above and \( \eta_t = w \) just below the interface.

### 4. The energy equation

Before finding a solution to the system of equations, let us consider the perturbation energy flux balance in the top layer which in dimensionless form can be written:

\[
\frac{\partial}{\partial t} \left\{ \int_{-1}^{0} \frac{1}{2} (v_1 \cdot \overline{v_1}) dz + \frac{1}{2} \alpha \eta^2 \right\} = R \eta_x \rho_1(-1) \\
+ w_z(-1)[p_z(-1) + \alpha \eta] - E \int_{-1}^{0} (u_{1z}^2 + v_{1z}^2) dz - \sigma \int_{-1}^{0} (u_{1x}^2 + v_{1x}^2) dz
\]

(2)

where subscripts 1 and 2 represent quantities in the upper and lower layers respectively and the overbars represent averaging over \( x \). The terms on the left are the rates of change of the kinetic energy density and the potential energy density of the interface. The first term on the right side represents energy flux from the mean flow, the second term the energy flux out of the upper layer (internal wave radiation) which is always negative. The remaining terms represent the frictional dissipation. The only source of energy into the perturbations is the mean flow, so that the only way there can be growth in the perturbations is for the energy from the mean flow to be greater than the sum of the rest of the terms taking energy out.

### 5. The stability analysis

Following the usual temporal stability analysis, we assume a solution of the form \( (u,v,w,p,\rho,\eta) = (\tilde{u},\tilde{v},\tilde{w},\tilde{p},\tilde{\rho},\tilde{\eta})e^{ikx-it\omega} \) and look for eigenvalues, \( \omega \), with positive imaginary parts which imply instability. In the mixed layer we find \( \tilde{\rho} = 0 \) and \( \tilde{\rho} \)
uniform in $z$. Because of this and the fact that stress vanishes at each boundary of
the mixed layer, there exist two infinite sets of eigenvalues which are completely
independent of all other conditions. These can be shown to be

$$\omega_{1,2} = Rk \pm f - i \left( \sigma k^2 + \frac{n^2 \pi^2 E}{2} \right), \quad n = 0, 1, 2, \ldots$$

(3)

and are clearly stable.

There are also a finite number of eigenvalues which do depend on the properties
of the bottom layer. For these roots, $\tilde{u}$ and $\tilde{v}$ in the mixed layer are uniform in $z$
and vertical stress does not enter. In this case we can show that in the mixed layer

$$\tilde{w} = -\frac{ik^2 Q\tilde{p}}{f^2 - Q^2} z,$$

(4)

where $Q = R k - \omega - i\sigma^2$. In the bottom layer we use the transform $\xi = \frac{1}{B_0} \int_{-1}^{z} B(z) dz$, where $B_0 = B(-1)$, as shown by Kroll. If we then assume $B(\xi) = B_0 e^{\gamma \xi}$, $\xi \leq 0$, where $\gamma$ is a constant, the equations will have constant coefficients and
can be easily solved, yielding in the bottom layer

$$\tilde{p} = -\alpha \eta + C_1 e^{(\gamma z + i\mu) \xi}$$

(5)

and

$$\tilde{w} = -\left( \mu - \frac{i\gamma}{2} \right) \frac{\omega C_1}{B_0^2} e^{-\left( \frac{\gamma}{2} \right)}$$

(6)

where $\mu = \sqrt{\frac{k^2 B_0^2}{(\omega^2 - f^2)} - \left( \frac{\gamma}{2} \right)^2}$ is the dimensionless vertical wave number and

$C_1$ is an arbitrary constant.

Here we must use the condition that the average vertical energy flux, $\overline{w(p + \alpha \eta)}$, in
the bottom layer must be downward to determine the proper branch for $\mu$. An
analysis of the vertical energy flux, averaged over one horizontal wavelength, shows
we must have $Re \left[ \omega \left( \mu - \frac{i}{2} \gamma \right) \right] \geq 0$.

In terms of $z$, we have $B(z) = B_0/(1 - \gamma(z+1))$, $z < -1$, which is fairly realistic.
The parameter $\gamma$ is a measure of the decrease of $N(z)$ with depth. For $\xi \to -\infty$, $\tilde{w}$ in
equation (6) could become unbounded. This is a consequence of dropping terms
small compared to $B^2$ which is invalid as $B \to 0$. In reality, if $B$ (i.e., $N$) becomes
sufficiently small, we will reach a vertical turning point. However, we are assuming
no perturbation sources in the lower layer so that it functions essentially as a bound-
ary condition to the upper layer. We are assuming that this boundary condition is
such that no energy from the upper layer is returned. Thus, if the point at which a
turning point could occur is sufficiently far from the interface, it will have little
effect on the role of the bottom layer as an energy absorbing boundary condition.
We are assuming $\omega = 0(f)$ which will insure that this turning point will not be near the interface.

Right now we are concentrating on a possible source of inertial motion and not on the details of its propagation in the deeper ocean. The proper analysis of deep ocean propagation would also have to include the latitudinal turning points on a $\beta$-plane (see Kroll). Clearly, for very shallow water, the assumption of no energy return is not a good one. But here we are not considering shallow water.

Using equations (4), (5), (6) and the kinematic conditions and the continuity of $\tilde{p}$ and $\tilde{\eta}$ at the interface, we obtain the eigenvalue equation

$$
 ik^{2}Q \left[ B_{o}^{2} + i \left( \mu - i \frac{\gamma}{2} \right) \alpha \right] - \left( \mu - i \frac{\gamma}{2} \right) \left( f^{2} - Q^{2} \right) (Rk - \omega) = 0 ,
$$

where $Q$ is defined in (4) and $\mu$ in (6). This result is actually the solution to a mathematically consistent model in that both the steady state and perturbation solutions are derivable from the same form of the equations of motion if $\nu_{o} \rightarrow \infty$. Any more realistic yet tractable model is unlikely to be mathematically consistent.

Equation (7) yields only two roots for $\omega$ that satisfy the radiation condition. One root seems to be always stable, but the other can be either stable or unstable, depending on $\alpha$, $B_{o}$, $\gamma$, $\sigma$, $R$ and $k$. We will call this root the unstable root. The general complex solution for $\omega$ must be found numerically. A solution for the neutral curve setting $\omega$ real is somewhat easier. The general method to do this for $\sigma \neq 0$ is described in Appendix 1. We will first consider some special cases.

6. Special cases

a. The case for $\sigma \rightarrow 0$. ($\nu_{o} \rightarrow 0$). As previously discussed, this model cannot have any physical validity unless there is decay ($\sigma \neq 0$). However, we can gain an insight into the mathematical structure of the general case by letting $\sigma \rightarrow 0$. If we separate equation 7 into real and imaginary parts, assuming $\omega$ real and then let $\sigma = 0$, we find four neutral curves assuming $\mu$ real and an infinite number for $\mu$ imaginary. On Figure 2 is shown a typical case on the $R, k$ plane of the nature of the unstable root for $\sigma = 0$ with dashed lines separating neutral, stable and unstable regions. All neutral solutions have no vertical energy flux.

The solid curves labeled one and two are two of the curves found for $\mu$ real. They are not special for the case $\sigma = 0$. However, for $\sigma$ slightly larger than zero, they become the only neutral curves present, separating the regions denoted by parentheses. Thus the neutral regions for $\sigma = 0$ immediately become either infinitesimally stable or unstable for an infinitesimal increase of $\sigma$ from zero. So the solid curves are of primary interest to us.
Figure 2. Typical neutral stability curves, Rossby number, $R$, versus dimensionless wave number, $k$, for $\sigma = 0$ (dashed lines) and $\sigma$ slightly greater than 0 (solid lines) for $\alpha = B_o = \gamma = 5$. Described in more detail in the text.

Curve one has $\omega = f$ and is given by

$$Rk = f + \sqrt{f^2 + \alpha k^2} \quad (8)$$

and curve two has $\omega = \left[f^2 + (2B_0k/\gamma)^2\right]^\frac{1}{2}$ and is given by

$$Rk = \left[f^2 + (2B_0k/\gamma)^2\right]^\frac{1}{2} + \left[f^2 + \frac{2k^2}{\gamma} \left(B_o^2 + \frac{\alpha\gamma}{2}\right)\right]^\frac{1}{2} \quad (9)$$

The asymptotes of these curves are shown on Figure 2. If $\gamma \to \infty$ or $B_o \to 0$ the curves become identical and the unstable region disappears. If $\gamma \to 0$ or $B_o \to \infty$, the asymptote of curve two goes to $R \to \infty$, and we have only one neutral curve.

Why does increasing $\gamma$ produce increasing stability? It would seem logical to first see how increasing $\gamma$ affects the vertical energy flux. If we use equations (5) and (6) in the vertical energy flux term from (2), we find that the downward flux at the interface is given by

$$-\frac{1}{2} \frac{|C_1|^2}{B_o^2} \text{Re} \left[ \omega \left(\mu_i + i\frac{\gamma}{2}\right)\right]$$

which for $\omega$ and $\mu$ real and positive can be shown to be

$$\frac{1}{2} \frac{\eta^2}{k^2} (\omega^2 - f^2) \omega \mu$$

in magnitude where $C_1$ was eliminated in favor of the magnitude of the interface displacement. Since $\mu^2 = k^2B_o^2/\omega^2 - f^2) - \gamma^2/4$, we see that the flux decreases for increasing $\gamma$. By itself this would imply decreasing stability. Thus increasing vertical energy flux is not the explanation.

Let us consider the energy flux from the mean flow given by

$$R \frac{\eta_p}{2} \frac{R\frac{\eta^2}{2}}{k} (\omega^2 - f^2) \text{Re} \mu.$$ 

The key is that there can be no energy to the perturbations unless
For this particular case, we have $\text{Re} \mu > 0$ only between curves (1) and (2) on Figure 2, which is consistent with the fact that this is the unstable region. The effect of increasing $\gamma$ is then to shut down the energy from the mean flow, stabilizing the flow.

If we let $\gamma \to 0$ and assume $\alpha k^2 << 1$, the remaining neutral curve is given by (8) and becomes $\text{Re} \mu > 0$. Defining a new Rossby type number $R = \frac{Uk}{f}$ (k dimensional), we must have $R < 2$ for stability. $R$ represents a ratio between inertia and Coriolis force which we must remember is valid as a measure of stability only for sufficiently large wavelengths.

We are interested in $\delta << 1$. But let us relax this condition and include the previously neglected nonhydrostatic terms. For $\gamma = 0$, neglecting vertical diffusion but retaining $\sigma$, equation (7) is modified to become $ik^2Q\frac{\tanh (a)}{a}[B_o^2 - \delta^2 \omega^2 + i\alpha \bar{\mu}] - \bar{\mu}(f^2 - Q^2)(Rk - \omega) = 0$ where $\bar{\mu}^2 = \frac{k^2(B_o^2 - \delta^2 \omega^2)}{\omega^2 - f^2}$ and $a = \frac{kQ \delta}{\sqrt{Q^2 - f^2}}$. In the limit $\sigma \to 0$ we still obtain neutral curve (1) given by (8) and another neutral curve given by $\omega = \frac{B_o}{\delta} = \frac{N_o}{f_o}$ and $Rk = \frac{B_o}{\delta} + \sqrt{f^2 + \alpha k^2}$. We can show the flow is stable everywhere except between the curves if $\frac{B_o}{\delta} > f$ (or dimensionally $N_o > f_o$).

For $B_o/\delta >> 1$, which is assumed, we essentially have the hydrostatic result. For $N_o \to f_o$, the instability disappears.

Still another neutral curve, independent of the previous two, is given by $\omega^2 = \frac{1}{2} \left[ f^2 + \frac{B_o^2}{\delta^2} + \sqrt{\left( \frac{B_o^2}{\delta^2} - f^2 \right)^2 + \frac{4 \alpha^2 k^2}{\delta^2}} \right]$ and $Rk = \omega$, where $R < \frac{\omega}{k}$ is stable and $R > \frac{\omega}{k}$ unstable. If we then let $f, B_o \to 0$, this solution goes into the limiting case for $\sigma \to 0$ of the Kelvin-Helmholtz instability. For $B_o/\delta >> 1$ the unstable region for this solution is outside our region of interest. That is for realistic values of $R$, the scales at which instability begins is on the order of the scale of the turbulence. Thus the solution we are considering is not a continuation of the solution for Kelvin-Helmholtz instability and does not behave as such.

b. The case for $\gamma = 0$. ($N$ constant). If we set $\gamma = 0$, equation (A1.2) in the Appendix 1 becomes quadratic in $\lambda^2$. Since we must have $\lambda^2 \geq 0$, the solution is

$$\lambda^2 = \frac{1}{2} \left[ f^2 + \alpha k^2 - \sigma k^4 + [(f^2 + \alpha k^2 - \sigma k^4)^2 + 4 \sigma^2 \alpha k^6]^{1/2} \right]. \quad (10)$$

We can then calculate $\mu$ from (A1.1), calculate $\omega$ from $\mu$ and finally calculate the neutral curve from $\lambda = Rk - \omega$. The expression is complicated and will not be reproduced. If we let $k$ become $>> 1$ we can show that $\omega \to f^2 + \frac{\sigma^2 k^4 \alpha}{B_o^2}$ and $R \to$
If we let \( k \to 0 \), we can show that \( \omega \to f \) and \( Rk \to 2 \). So we know that \( R \) must have a minimum for some \( k \) since \( R \to \infty \) for \( k \to \infty \) and \( k \to 0 \). An example of this case is shown on Figure 4. The point on the neutral curve where \( R \) is a minimum is called the critical point \((R_c, k_c)\).

c. The case for \( \alpha = 0 \) \((\Delta \rho = 0)\). If we let \( \alpha = 0 \) in the \( \gamma = 0 \) case we see that \( R \to 0 \) as \( k \to \infty \) implying short wavelength catastrophe. So our model is really not valid for both \( \alpha = 0 \) and \( \gamma = 0 \). However, for \( \gamma \neq 0 \) and \( \alpha = 0 \) a minimum value for \( R \) at a finite value of \( k \) is realized analogous to the behavior if the two curves (8) and (9) for \( \sigma = 0 \) could intersect at some finite \( R \).

If we let \( \alpha = 0 \) in (A1.2), we will find three possible solutions in \( \lambda^2 \). The first two are given by

\[
\lambda^2 = f^2 + k^2B_o^2/\gamma - \sigma^2k^4 = k^2 \left[B_o^2/\gamma^2 - 4f^2\sigma^2\right]^{\frac{1}{2}}
\]  

and the third by \( \lambda^2 = 0 \). Associated with each of these solutions is a neutral curve. We will denote the curves associated with (11) as one (top sign) and two (bottom sign) respectively and that for \( \lambda^2 = 0 \) as curve three. The first two are algebraically complicated and not reproduced here. The third is simply \( Rk = f \) with \( \omega = f \) and is a neutral curve for all \( k \geq 0 \). The other two curves are valid only for values of \( k \) such that \( \lambda^2 > 0 \).

These three curves bound the unstable region as shown for a typical case on Figure 3. The dashed extension of curve three is neutral but the solution is stable on both sides. The critical value of \( k \) is where \( \lambda^2 = 0 \) for curve one from (11). If \( \gamma \to 0 \), curve one goes off to infinity and we have the case \( \alpha = 0, \gamma = 0 \) described previously.

An important feature to note is that if the quantity under the radical in (11) goes to zero, curves one and two are identical, and there can be no interior unstable region. For \( B_o^4/\gamma^2 < 4f^2\sigma^2 \) curves one and two cease to exist and we have stability everywhere except on the neutral curve \( Rk = f \). In Appendix 1 we show that in general, not just for \( \alpha = 0 \), there can be no instability unless

\[
B_o^2 > 2f\gamma \sigma.
\]

This is analogous to the \( \sigma = 0 \) case where \( \gamma \to \infty \) or \( B_o \to 0 \) produces the same effect. The explanation is essentially the same. The energy from the mean flow decreases with decreasing \( Re\mu \) which decreases with increasing \( \gamma \). The balance between stability and instability is even more sensitive to this than for the \( \sigma = 0 \) case.

7. General solution and discussion

Using the method outlined in Appendix 1, we can calculate the neutral curves
Figure 3. Typical neutral stability curves for \( \sigma = 1 \), \( B_0 = 2 \), \( \gamma = 1 \) and various values of \( \alpha \) increasing from 0. The small circle marks the critical point of the particular curve and the number beside it the frequency at the critical point. The unstable region is inside the particular curve, the stable outside. Described in more detail in the text.

for any values of the parameters. On Figure 3, besides the \( \alpha = 0 \) case, we show typical neutral curves for \( \alpha = 0.01, 0.1, \) and \( 0.5 \), indicating increasing stability with increasing \( \alpha \). This we would expect since this means that the stabilizing density jump across the interface is increasing. The critical point on each curve is denoted by a small circle with the adjacent number the frequency at the critical point. This frequency increases with \( \alpha \).

Figure 4 also shows some typical neutral curves. Curve one is for \( \gamma = 0 \) and shows the linear increase of \( k \) with large \( k \) as described previously. The other two curves for \( \gamma = 1 \) show the effect of decreasing \( B_0 \). If we made \( B_0 < \sqrt{2} \), the neutral curve would disappear and the flow is completely stable as predicted by the inequality (12).

We have been using \( \sigma = 1 \) except for the \( \sigma = 0 \) case which implies we are assuming \( L = \sqrt{v_h/f_0} \). This makes sense since we know that our results are invalid unless the horizontal decay makes an \( O(1) \) contribution which choosing this horizontal scale ensures. So we will assume \( \sigma = 1 \) and \( L = \sqrt{v_h/f_0} \) from now on.

On Figure 5 we show stable and unstable curves as well as the neutral curve for a typical case. The parameter denoting the curves is the imaginary part of \( \omega \) where positive values are of course unstable and negative stable. It is important to note that the growth rate of the instability is limited for \( \gamma > 0 \). If \( B_o^2 - 2f\sigma\gamma > 0 \) but approaching zero, the maximum growth rate approaches zero.
We can estimate the magnitude of $\sqrt{\nu h/f_o}$ necessary to produce instability using the inequality (12). Realistically we have $h = 0(10 \text{ m})$, $N_o/f_o = 0(100)$, $f_o = 0(10^{-4} \text{sec}^{-1})$ and $\gamma = 0(1)$ where $N_o$ is the value of $N$ just below the interface.

From (12) we must have $B_o \geq 0(1)$ for possible instability. Since $B_o = \frac{h N_o}{L f_o}$, we must then have $\sqrt{\nu h/f_o} \leq 0(1 \text{ km})$. This in turn implies that $\nu h \leq 0(10^6 \text{cm}^2/\text{sec})$. Thus, though we must have significant lateral friction, to have possible instability it cannot be too large.

Ballpark estimates of the parameters give limited information because the question of stability of the system is quite sensitive to the values of the parameters. It is of course sensitive to the value of $R$ but also to the depth of the mixed layer $h$ through the inequality (12). If we visualize the mixed layer deepening, the parameters will in general change. In Appendix 2, assuming our model with no heat transfer, we calculate the increase of $\Delta \rho$ with $h$ and then calculate $\alpha$, $B_o$ and $\gamma$ in terms of $h_0$, the initial depth before deepening, $N_o^0$, the initial $N_o$, $\gamma$, the dimensional equivalent of $\gamma$, and $\Delta \rho_o$, the initial density jump, as well as $h$ and $L$. Since our calculations have been based on constant $h$, we are of course assuming that $h$ is changing slowly with respect to our characteristic time scale which is the inertial period. This is certainly not true for the onset of deepening from $h = 0$ produced by a strong wind.
Figure 5. Typical stable and unstable curves for $\alpha = 0.5, B_0 = 2.0, \gamma = 1.0$ and $\sigma = 1.0$. The parameter is $\text{Im}(\omega)$.

8. Two differing unstable flow scenarios

The results of our analysis suggest two basically different flow scenarios which can result in an instability. The first we call the "increasing flow rate instability." It is the usual type of instability occurring in parallel flow. For $h$ essentially fixed, we visualize the flow rate increasing until a critical value of $R$ is reached, if ever. The second we call the "increasing depth instability" which is essentially determined by (12). Here we visualize the flow rate as essentially fixed and the depth of the mixed layer gradually increasing until a condition of instability is reached, if ever.

a. Increasing flow rate instability. We visualize that initially the flow rate in the mixed layer of fixed depth $h$ is subcritical and is increasing. Figure 6 represents typically such a flow where critical values of the dimensionless flow rate, $R\delta_c$, and the wave number, $k_c$, are plotted versus $L = \sqrt{\nu_h/\nu_o}$. In this particular case we have $h = 10$ m, and $\Delta \rho = 0$, and reasonable values of $\hat{\gamma}^{-1} = 20$ m and $\frac{N_0}{\nu_o} = 40$ can be estimated from Site D data (Pollard). The critical value of the frequency, $\omega_c$, is exactly 1.0 for all values of $\sqrt{\nu_h/\nu_o}$, a consequence of the fact that $\Delta \rho = 0$ ($\alpha = 0$) (see Fig. 3).

We can show for $\sqrt{\nu_h/\nu_o} \geq \frac{N_o}{\nu_o} \sqrt{\nu_o/2\hat{\gamma}}$ that the inequality (12) is no longer true and the system is stable. For the values of the parameters used, this corresponds to $\sqrt{\nu_h/\nu_o} \geq 400$ m, and the wavelength, $\frac{2\pi}{k_c} \sqrt{\nu_h/\nu_o}$, is a maximum of about 1.6 km.
Figure 6. Dimensionless critical wave numbers, $k_c$, and flow rate, $R\delta_c$, versus $L = \sqrt{\nu_h/f_o}$ for typical conditions for the increasing flow rate instability scenario where $\alpha_e = 0$, $h_e = 10$ m, $\gamma^{-1} = 20$ m. For the critical frequency we have $\omega_e = 1$ for all $L$. Parameters defined and the figure described in more detail in the text.

The actual flow rate is given by $hU = (R\delta)v_h$ and is also a maximum of about .26 m²/sec at $\sqrt{\nu_h/f_o} = 400$ m. This corresponds to $U \approx 3 \text{ cm sec}^{-1}$ for $h = 10$ m, a quite realizable flow.

As shown in Figure 7, typically the critical frequency increases very fast with an increase in $\Delta \rho$ ($g' = g\Delta \rho/\rho$). In fact for $g' > .01$ the frequency really can no longer be called near-inertial. As expected the system becomes more stable ($R\delta_e$ increases) as $g'$ increases. Another quantity shown on the figure is $r$, the ratio of the energy density below to that above the interface. It approaches $\infty$ for $g' \rightarrow 0$ and decreases quickly with $g'$ but is substantially greater than one for most of the realistic values of $g'$ which are from 0 to about .2 cm/sec².

For this instability to be physically realizable, some special conditions must be fulfilled. First the mixed layer must be established previous to any substantial acceleration of the flow rate. This might occur due to heating or a previous wind event. Secondly the depth of the mixed layer must be sufficiently deep such that the
inequality (12) is satisfied. For the calculations leading to Figures 6 and 7 to be valid we must also have no deepening as the flow rate accelerates.

If the layer does deepen from the initial state, we must use the results of Appendix 2 to calculate the change in the parameters with deepening. Figure 8 shows a typical case where the layer deepens from 10 m. The critical flow rate increases and especially the frequency due to the increase in $\alpha$ which is also shown on the

Figure 7. Dimensionless critical flow rate, $R\delta_c$, frequency, $\omega_c$, wave number, $k_c$, and the ratio of the energy density below to that above the interface, $r$, versus the reduced gravity $g' = \frac{\Delta \rho}{\rho} g$, for the increasing flow rate scenario. The typical conditions are $h_o = 10 \text{ m}$, $N_o/f_o = 40$ and $\sqrt{\nu}/f_o = 20 \text{ m}$ with $L = \sqrt{\nu}/f_o = 300 \text{ m}$.

Figure 8. $\alpha$, dimensionless critical flow rate, $R\delta_c$, frequency, $\omega_c$, and wave number, $k_c$, versus depth increasing from $h_o$ for the increasing flow rate scenario. The typical conditions are the same as for Figure 7 except $L = 230 \text{ m}$. 
Figure 9. Typical neutral stability curves to illustrate the increasing depth scenario. All curves have $\alpha = \frac{1}{2} B_0^2$, $\gamma = 2.0$ and $\sigma = 1$. $B_0$ is: 2.01 for curve 1, 2.3 for curve 2, 2.67 for curve 3, and 5.0 for curve 4.

Assuming that the flow accelerated faster than the layer deepened, instability would occur when the flow reached critical for some particular depth $\geq 10$ m.

b. *Increasing depth instability.* Here we visualize the flow rate of the mixed layer constant and the depth initially subcritical (i.e., inequality (12) not fulfilled) and gradually increasing. To get an idea of what takes place consider Figure 9. Here we plot a succession of neutral curves for $B_0$ above the critical value which is $B_0 =$
2 for $\gamma$ fixed at 2. We neglect any change in $\gamma$ or $N_o$ with depth, but assume $\alpha = \frac{1}{2} B_o^2$ from (A2.7) for $\alpha_o = B_o^o = 0$, and we note that $B_o = \frac{N_o}{f_o} \frac{h}{L}$ with $L$ fixed. We see from the figure that as $B_o$ increases ($h$ increases) from 2.0 the critical flow, $R_c$, decreases until $R_c$ reaches a minimum for $B_o \approx 2.67$ and then increases with increasing $B_o$. This type of behavior would be expected from a balance between increasing instability from an increasing $B_o$ and increasing stability from an increasing $\alpha$. For a given $R_c$ greater than the minimum there are two possible values of $B_o$, but only the value associated with the shallower depth would make physical sense.

We learn from the behavior of the neutral curves in Figure 9 that, not only is there a value of $B_o$ below which the system is stable for all conditions, there is a value of $R$ below which the same is true. This is shown in a more extensive calculation represented typically by Figure 10. Here we assume that there is no initial depth for the mixed layer ($h_0 = 0$) and that it gradually deepens. The calculations in Appendix 2 are used to calculate the changes of the parameters with depth. In general we can show, using the inequality (12), that this instability cannot occur unless $\sqrt{h_0/2\bar{\gamma}} \left(\frac{N_o^o}{f_o}\right) < \sqrt{\nu_h/f_o} < \frac{1}{\bar{\gamma} \sqrt{2}} \left(\frac{N_o^o}{f_o}\right)$ which also implies we must have $h_0 < \bar{\gamma}^{-1}$. Assuming again that $\bar{\gamma}^{-1} = 20$ m and $N_o^o/f_o = 40$, the maximum value of $\sqrt{\nu_h/f_o}$ is 566 m. Using $\sqrt{\nu_h/f_o} = 300$ m in the figure, we see that if the dimensionless flow rate, $R\bar{\delta}$, is less than about .14, an unstable state will never be reached. If $R\bar{\delta} > .14$ instability will not occur unless $h$ reaches 8 m except in a small interval
.14 \leq R \delta \leq .2 where a larger depth must be reached.

For a realistic flow rate of say 5 m$^2$/sec, we would have $R \delta \approx .55$ for $\sqrt{v_h/f_o} = 300$ m which from Figure 10 implies that the critical depth ($h_c$) is 8 m, $\omega_c \approx 1.06$ and $k_c \approx .1$. The horizontal wavelength would then be $L_x = \frac{2\pi}{k_c} \sqrt{v_h/f_o} \approx 19$ km.

If for this same flow rate we have $\sqrt{v_h/f_o} = 230$ m then $h_c = 4$ m, $\omega_c \approx 1.015$ and $L_x \approx 40$ km. If $\sqrt{v_h/f_o} = 460$ m, the flow rate is not sufficiently large to produce an instability.

c. Comparison of the flow scenarios. The nature of the instability for the "increasing flow rate" (first scenario) and "increasing depth" (second scenario) contrast sharply in some aspects. The horizontal wavelength of the first is in general $0(1 \text{ km})$ while that of the second is $0(10 \text{ km})$. This will carry over to the vertical wavelength where that for the first ($0(10 \text{ m})$) will be much less than that for the second ($0(100 \text{ m})$) for the same near-inertial frequency. These are related through the equation for the dimensionless wave number $\mu$ from (6). The ratio of the energy densities below and above the interface, $\rho$, is $>> 1$ for the first and $< 1$ for the second. The minimum flow rate for instability is generally much less for the first than the second.

An important observation is that it seems less likely for the first scenario than the second that the frequency of the instability will be near-inertial (within about 15% of $f_o$). Figure 7 illustrates that $\omega_c$ for the first is near-inertial only for $g'$ very small; i.e., essentially no density jump. Figure 10 illustrates that $\omega_c$ for the second will be near-inertial for all possible values of $R \delta$ except in a small interval near the minimum value of $R \delta$. Hence it appears that if there is an instability in the second scenario, it will probably produce near-inertial motion.

9. Conclusion

The model is crude and quite sensitive to the parameters involved. Especially unsettling is the dependence of the nature of the instability on the unknown horizontal eddy viscosity. But the results do indicate that the mean flow of the mixed layer could become unstable on a horizontal scale much larger than its depth. The frequency would likely be near-inertial so that the instability could be a source of the inertial motion observed down through the water column. This source would not depend on the horizontal variability of the wind to produce Ekman suction to in turn generate downward propagating inertial motion.

If the instability actually exists, the question is then how one would detect it. Certainly, if one observed little wind-generated inertial activity in the mixed layer but substantial inertial motion with downward group velocity beneath the mixed layer, this would be strong evidence. But this is quite unlikely since there is almost always significant inertial motion in the mixed layer directly forced by the wind. Still, even with strong directly forced inertial motion in the mixed layer, the ob-
servation of downward propagating oscillations concentrated about a wavelength unrelated to the wind forcing would also be indicative.

A salient feature of the instability is scale. Our calculations show that the energy from the instability should be concentrated about a horizontal wavelength \( \leq 0(10 \text{ km}) \). Sanford observed this scale indirectly, using the dispersion relation and his data for the vertical spectrum. A direct observation of horizontal scale beneath the thermocline would certainly be desirable. Pollard did investigate directly the horizontal scale at Site D beneath the mixed layer and found no coherence for three moorings separated by 50 and 70 km. Our results indicate this spacing may be too great for significant coherence. If the instability exists, the oscillations from the directly forced problem will also favor the scale of the instability. Thus the definite determination that there is a favored horizontal scale unrelated to the forcing scale would be strong evidence.

A more realistic model would certainly contain some shear in the mixed layer especially some shear normal to the transport direction. This would then allow the inviscid instability, which our model has suppressed, to enter the system. Also the inertial oscillations directly forced by the wind in the mixed layer should be considered within the steady state solution. However, it would not seem worthwhile to consider a model necessitating extensive numerical analysis until some fairly strong observational evidence of an instability is found.

We have not addressed the question of whether the instability can significantly affect the energy balance of the mixed layer and thus affect the deepening. This is because our stability analysis does not give us amplitude. All we can say is that the growth of the instability is bounded with the maximum value of \( \text{Im}(\omega) = 0 \left( \frac{1}{10} \right) \) which translates as an increase of amplitude by a factor \( e \) in a couple of days, not an explosive instability.

**Acknowledgments.** This work was supported by the National Science Foundation under grant number OCE 8006048.

**APPENDIX 1**

The neutral stability curve

We let \( \omega \) be real in (7) and separate into real and imaginary parts. From the imaginary part we obtain

\[
\mu = \frac{\lambda}{\sigma k^2 (2\lambda^2 - \alpha k^2)} \left[ k^2 \left( B + \frac{\gamma}{2} \alpha \right) + \frac{\gamma}{2} \left( f_0^2 - \lambda^2 + \sigma^2 k^4 \right) \right]
\]

where \( \lambda = R k - \omega \), and \( \mu \) is defined after (6). If then \( \mu \) is eliminated in the real part using (A1.1), we obtain

\[
\lambda^8 + A_2 \lambda^4 + A_4 \lambda^2 + A_6 = 0
\]

and the coefficients can be written in the form: \( A_6 = a \sigma^2 k^6 (2B + a) \), \( A_4 = q^2 - k^4 r \) and \( A_2 = \))
\[-2q, \text{ where } B' = \frac{B_o^2}{\gamma}, q = f^2 - \sigma^2 k^4 + k^2(B' + \alpha), \text{ and } r = B'^2 - 4f^2\sigma^2. \]

For fixed values of $B_o, \gamma$ and $\sigma$, a neutral curve can be found by first choosing a value for $k$ and solving the cubic equation (A1.2) for $\lambda^2$. Then we use (A1.1) to find $\mu$ and the definition of $\mu$ to find $\omega$. We can then find the value for $R$ from the definition $\lambda = Rk - \omega$. For a given $k$ you obtain two valid solutions for $\lambda^2$ and hence two values of $R$ unless $\gamma = 0$. For $\gamma = 0$, (A1.2) becomes quadratic in $\lambda^2$ and there is only one valid value of $R$.

### Asymptotic and other special properties

The nature of the solution of (A1.2) is determined by the character of the discriminant,

\[ D = \frac{b^2}{4} + \frac{a^3}{27} \quad (A1.3) \]

where $a = A_1 - \frac{1}{3}A_2^2$ and $b = A_0 + \frac{2}{27}A_2^3 - \frac{1}{3}A_1A_2$. Only solutions for which $D \leq 0$ can be a neutral curve. If we substitute for $a, b, A_0, A_1$ and $A_2$, we find

\[ D = k^2 \left\{ \frac{1}{4} \alpha \sigma^2 C \left( \frac{1}{27} q^3 \right) - r \left[ \frac{1}{27} \left( q^2 + 2rk^2q + k^2r^2 \right) + \frac{\sigma^2k^3}{3} \right] \right\} \quad (A1.4) \]

where $C = 2B' + \alpha$. From this equation we can see that if $r < 0$, we have $D > 0$ for $k \geq 0$ if $q > 0$ and hence there is no neutral curve. If $q < 0$ and sufficiently large enough in magnitude to make $D \leq 0$, then we obtain $\lambda^2 < 0$ which again means no neutral curve. So we must have $r > 0$ to have any unstable solutions which is equivalent to the expression (12). We note also that for $r \to 0$ and $\alpha \neq 0$ we must have $k \to 0$ to maintain $D \leq 0$.

If we let $k \to 0$ in (A1.2) we find that $\lambda \to f + \frac{1}{2} k^2 \frac{1}{f} (B' + \alpha - \sqrt{r})$. If we let $k \to 0$ in (A1.1) we then find $\mu \to \gamma(B' + \sqrt{r})/4\sigma f$, and, from the definition of $\mu$, that

\[ \omega \to f. \quad (A1.5) \]

Then from the definition of $\lambda$, we find that as $k \to 0$,

\[ Rk \to 2f. \quad (A1.6) \]

This implies that the neutral curve for $r \to 0$ is the point at which $k = 0$ and $R \to \infty$ such that $kR = 2$.

### APPENDIX 2

#### The variation of the parameters for a deepening mixed layer

We assume the variation of $N$ with $z$ in the lower layer in dimensional terms is given by

\[ N = \frac{N_o^o}{1 - \gamma(z + h_o)} , \quad z \leq -h_o, \quad (A2.1) \]

for the initial state before any deepening of the mixed layer. Here $N_o^o$ is the value of $N$ at the initial depth of the mixed layer, $h_o$, and $\gamma$ is the dimensional equivalent to $\gamma$. The density at any $z$ can be calculated from (A2.1) since $N^z = \frac{-g}{(\rho_o + \Delta\rho_o) \frac{dp}{dz}}$ where $\rho_o$ is the initial uniform density in the mixed layer and $\Delta\rho_o$ the initial density jump. Thus

\[ \rho = \rho_o + \Delta\rho_o + \frac{(\rho_o + \Delta\rho_o)N_o^o}{g\tilde{\gamma}} \left[ 1 - \frac{1}{1 - \gamma(z + h_o)} \right]. \quad (A2.2) \]

We now assume that the mixed layer mixes down from $h_o$ to $h$. If we assume no transfer of heat out of the system, then the new uniform density will be the average density of the fluid above $z = -h$ given by
\[ \tilde{\rho} = \frac{1}{h} \left[ \rho_0 h_0 + \int_{-h_0}^{h_0} \rho dz \right]. \]  
(A2.3)

The new density jump is then given by

\[ \Delta \rho = \rho(-h) - \tilde{\rho} = \frac{(\rho_0 + \Delta \rho_o)N_o^2}{g\gamma h} \left[ h_0 - \frac{1}{1 + \tilde{\gamma}(h - h_0)} + \frac{1}{\gamma} \ln(1 + \tilde{\gamma}(h - h_0)) \right] + \frac{h_0}{h} \Delta \rho_o \]  
(A2.4)

Using the definition of the parameters \( \alpha \) and \( B \), we find

\[ \alpha = \tilde{B} \{ B_o^* - B_o + \tilde{B} \ln(1 + \tilde{\gamma}(h - h_0)) \} + \alpha_o \]  
(A2.5)

where

\[ \tilde{B} = \frac{N_o^2}{f_o L \gamma}, \quad B_o^* = \frac{N_o^2 h_0}{f_o L}, \quad B_o = \frac{N_o^2 h}{f_o L \left[ 1 + \gamma(h - h_0) \right]} \]

(the value of \( B \) at \( h \)), and \( \alpha_o = \frac{g \Delta \rho_0 h_o}{(\rho_0 + \Delta \rho_o)^2 f_o^2 L^2} \) (the initial value of \( \alpha \)). We can also show that the dimensionless \( \gamma \) we are using will change with the deepening of the mixed layer and is given by

\[ \gamma = \frac{\tilde{\gamma} h}{1 + \tilde{\gamma}(h - h_0)} . \]  
(A2.6)

For \( \tilde{\gamma}(h - h_0) << 1 \), we can show that

\[ \alpha \equiv \frac{1}{2} \left( B_o^2 - B_o^* \right) + \alpha_o \]  
(A2.7)

and \( \gamma \equiv \tilde{\gamma} h \) and \( B_o \equiv N_o^2 h/f_o L \).

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Received: 15 July, 1980; revised: 17 June, 1982.