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On the advective model of the thermocline circulation

by P. F. Hodnett

ABSTRACT

A theoretical study is made of the motion in the thermocline region of a large ocean. It is assumed that motion takes place under geostrophic and hydrostatic balance and that heat is transferred by convection only. Proper account is taken of the earth's spherical geometry. A class of solutions of the mathematical problem is derived and one of these, an inverse power solution, is discussed in detail. The solution does not satisfy all the details of the known physical situation but shows reasonable qualitative agreement with it. In particular the solution exhibits westerly surface flow at high latitudes and easterly surface flow at low latitudes. It also shows equatorward flow in the western ocean in the region beneath the surface Ekman layer.

1. Introduction

In the recent past there have been a number of theoretical analyses of the dynamics of the thermocline region of large oceans. Welander (1959) considered an advective model of the thermocline where transfer of heat is by convection only, the Boussinesq approximation is employed and conservation of momentum is expressed through geostrophic and hydrostatic balance. His solution explained some but not all of the physically observed features of motion in the thermocline. At the same time Robinson and Stommel (1959) used a different model in considering the thermocline; they included vertical diffusion of heat but neglected heat convection in one of the horizontal directions. In a subsequent investigation Robinson and Welander (1963) included all the heat convection terms together with vertical diffusion of heat in a model which led to a more complicated mathematical problem for which certain restricted solutions were derived. Blandford (1965) presented an analytical solution for the model proposed by Robinson and Welander and this solution is partially successful in explaining the physical phenomena. The same model, which includes vertical diffusion of heat, was used by Needler (1967) in his analysis of the problem. These papers were comprehensively reviewed by Veronis (1969) who analyzed the relationships between the different solutions. He also produced some general solutions of which two particular cases are the inverse...
power solution of Fofonoff (1962) and the exponential solution of Blandford (1965).

In this paper the motion in the thermocline is considered to be described by the advective model as in Welander (1959). This solution for the thermocline must then be matched to the solution for the surface Ekman layer and to solutions for the ocean side boundary layers in order to describe the total motion in large oceans. In this paper a class of solutions of the mathematical problem are derived one of which is the exponential solution of Welander (1959). The other solution of the class which is discussed in detail here is an inverse power solution. This solution is not related to the inverse power solution of Fofonoff since that entire solution vanishes when vertical diffusion of heat is zero which is one of the assumptions under which the solution presented in this paper is derived.

2. Formulation of the problem

The thermocline region below the surface Ekman layer and above the cold layer of water on the ocean floor is viewed in this paper as distant from the side boundaries of the ocean. Then it is reasonable to argue (along the lines detailed by Welander, 1959) that viscosity and heat diffusion are negligible and that inertial acceleration is negligible in comparison to Coriolis acceleration. Then motion in the thermocline is determined by the equations which express geostrophic and hydrostatic balance for the conservation of momentum. These are supplemented by the equations for conservation of mass and energy. Given that there are small variations in density, the Boussinesq approximation is used and for a shallow ocean the equations written in spherical coordinates are then those given in Welander (1959) (eqns. (2') through (6')) and Veronis, 1969, (eqns. (2.1) through (2.5) with $K = 0$. $K$ is the heat diffusion coefficient).

With primed quantities denoting physical variables, the pressure $p'$, is written as $p' = - \rho_0'gz' + \overline{p}$ where $\rho_0'$ is the reference density, $g$ is gravity and $z'$ is the vertical coordinate, positive upwards. The shallow ocean approximation is used where the radial distance $r' = a + z'$ is replaced by $a$ where $a$ is the earth's radius. A nondimensional temperature $\bar{T}$ is introduced where $\bar{T} = (t'-t_0')/t_0'$ and $t_0'$ is the reference temperature. Recognizing that the depth of the thermocline (of order $2 \times 10^5$ cm.) is of order of magnitude different from the earth's radius, $a$ ($a = 6.3 \times 10^8$ cm.) and that the vertical velocity component, $w'$ (of order $5 \times 10^{-5}$ cm/sec according to Montgomery, 1936) is of order of magnitude different from the horizontal components of velocity $u'$, $v'$ (both of order 1cm/sec) we nondimensionalize as follows $u' = Uu$, $Uv$; $w' = Ww$; $z' = hz$; $\overline{p} = (\rho_0' a \Omega U) \overline{p}$ where $\Omega$ is the rate of rotation of the earth. Then the following nondimensional equations result

$$n_1 \frac{\partial \bar{p}}{\partial z} = \bar{T}, 2u \sin \phi = - \frac{\partial \bar{p}}{\partial \phi}, 2v \sin \phi = \frac{1}{\cos \phi} \frac{\partial \bar{p}}{\partial \lambda}, \quad (1)$$
\[ n_2 \frac{\partial w}{\partial z} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} (v \cos \phi) + \frac{1}{\cos \phi} \frac{\partial u}{\partial \lambda} = 0, \]  
(2)

\[ n_2 \ w \frac{\partial T}{\partial z} + v \frac{\partial T}{\partial \phi} + \frac{u}{\cos \phi} \frac{\partial T}{\partial \lambda} = 0, \]  
(3)

where \( \lambda \) (longitude, positive eastwards), \( \phi \) (latitude, positive northwards) are coordinates and the nondimensional velocity components in these directions are \( u, v \) respectively. The nondimensional numbers \( n_1, n_2 \) are given by:

\[ n_1 = a \Omega U / (h g a t_o'), \]

\[ n_2 = a W / (h U) \]

where \( a \) is the coefficient of thermal expansion. With \( a = 6.3 \times 10^8 \) cm, \( \Omega = 0.7 \times 10^{-4} \) (sec\(^{-1}\)), \( U = 1 \) cm/sec, \( h = 2 \times 10^5 \) cm, \( g = 981 \) cm/sec\(^2\), \( \alpha = 2 \times 10^{-4} \) (°C\(^{-1}\)), \( t_o' = 4 \)°C (\( t_o' \) is taken to be the temperature at the bottom of the thermocline) we obtain \( n_1 = 0.281 \). With \( W = 5 \times 10^{-5} \) cm/sec (Montgomery, 1936, indicates that this is a typical size for the vertical velocity at the bottom of the Ekman layer) we deduce \( n_2 = 0.1575 \).

These equations are simplified by putting

\[ z_1 = z/n_1, \ u_1 = \frac{\sin^2 \phi}{\cos \phi} u, \ v_1 = \sin \phi \cos \phi v, \ w_1 = (n_2/n_1) \sin^2 \phi \ w \]

and following Welander (1959) the introduction of a new coordinate \( \eta = \log \sin \phi \) gives

\[ \frac{\partial p}{\partial z_1} = T, \ u_1 = -\frac{1}{2} \frac{\partial p}{\partial \eta}, \ v_1 = \frac{1}{2} \frac{\partial p}{\partial \lambda}, \]  
(4)

\[ \frac{\partial w_1}{\partial z_1} + \frac{\partial v_1}{\partial \eta} - v_1 + \frac{\partial u_1}{\partial \lambda} = 0, \]  
(5)

\[ w_1 \frac{\partial T}{\partial z_1} + v_1 \frac{\partial T}{\partial \eta} + u_1 \frac{\partial T}{\partial \lambda} = 0. \]  
(6)

We note that equations (4) imply

\[ \frac{\partial u_1}{\partial \lambda} = -\frac{\partial v_1}{\partial \eta} \]

so that two terms of the continuity equation (5) vanish and we are left with the simpler planetary divergence relation:

\[ \frac{\partial w_1}{\partial z_1} - v_1 = 0. \]  
(5a)

If the substitution \( T = \frac{\partial^2 M}{\partial z_1^2} \) is made in equations (4), (5a), (6) above then the \( M \)-equation of Welander (1959) is produced. Welander was able to derive one solution of the \( M \)-equation which represents, at least qualitatively, some aspects of the known physical situation. By reformulation of the problem as defined by equations (4), (5a), (6) this paper presents a class of solutions of these equations (one solution is Welander's 1959 solution) which are capable of representing some but not all characteristics of the known physical situation.

This is achieved by interchanging the roles of the independent variable \( z_1 \) and the dependent variable \( T \) by now regarding the nondimensionalized perturbation temperature \( T \) as an independent variable and the nondimensionalized vertical
height \( z \) as a dependent variable. Under this transformation the independent variables are \((\lambda, \eta, T)\) and then equations (4), (5a), (6) take the form

\[
\frac{\partial p}{\partial T} = T \frac{\partial z_1}{\partial T}, \quad u_1 = -\frac{1}{2} \left( \frac{\partial p}{\partial \eta} - T \frac{\partial z_1}{\partial \eta} \right), \quad v_1 = \frac{1}{2} \left( \frac{\partial p}{\partial \lambda} - T \frac{\partial z_1}{\partial \lambda} \right),
\]

(7)

\[
\frac{\partial w_1}{\partial T} - v_1 \frac{\partial z_1}{\partial T} = 0,
\]

(8)

\[
w_1 - v_1 \frac{\partial z_1}{\partial \eta} - u_1 \frac{\partial z_1}{\partial \lambda} = 0.
\]

(9)

Putting \( G = p - Tz_1 \) gives

\[
z_1 = -\frac{\partial G}{\partial T}, \quad u_1 = -\frac{1}{2} \frac{\partial G}{\partial \eta}, \quad v_1 = \frac{1}{2} \frac{\partial G}{\partial \lambda}
\]

(10)

together with equations (8) and (9).

Equation (10) gives \( z_1, u_1, v_1 \), as functions of \( G \). Then substitution in equation (9) for \( z_1, u_1, v_1 \) gives \( w_1 \) as a function of \( G \). Finally, substitution in equation (8) for \( w_1, v_1 \) and \( z_1 \) gives the following partial differential equation for \( G \):

\[
\lambda \, G_{TT} + \eta \, G_{T\eta} - G_\lambda \, G_{T\lambda} = 0.
\]

(11)

The \( G \)-equation (11) is a simpler equation than the \( M \)-equation used in Welander.

### 3. Particular solution of the \( G \)-equation

It has not been possible to derive the general solution of equation (11). However it is apparent how to find certain particular solutions of the equation.

An obvious solution of (11) is \( G_\lambda = 0 \). In this case equations (10) and (9) imply that \( v_1 = w_1 = 0 \). This solution (also noted by Welander) then corresponds to purely zonal variations of the remaining nonzero variables, i.e. \( u_1, p, T \), and it would be necessary to append meridional boundary layers.

The most general solution found is that given by splitting equation (11) into

\[
G_\eta \, G_{TT\lambda} = 0,
\]

(12)

and

\[
G_\lambda \, (G_{TT} - G_{T\eta}) = 0,
\]

(13)

and taking \( G_{TT\lambda} = 0 \) and \( G_{TT} - G_{T\eta} = 0 \) as the solutions of (12) and (13) respectively. The other possible solution i.e. \( G_\eta = 0 \) and \( G_{TT} = 0 \) is not physically realistic.

The solution of \( G_{TT\lambda} = 0 \) is \( G_{TT} = H(\eta, T) \), where \( H \) is an arbitrary function of \( \eta \) and \( T \).

Then the equation \( G_{TT} - G_{T\eta} = 0 \) becomes

2. The author is grateful to Professor Claes Rooth for the suggestion that this transformation might simplify the problem.
The solution of this equation is

$$H(\eta, T) = K(T) e^{\eta},$$

where $K$ is an arbitrary function of $T$. We then have

$$G_{TT} = K(T) e^{\eta},$$

whose solution is

$$G = e^{\eta} N(T) + T S(\lambda, \eta) + L(\lambda, \eta) \quad (14)$$

where $N$, $S$, $L$, represent arbitrary functions.

For easier association with the physical situation we now replace the variable $\eta$ by the physical coordinate $\phi$ (note $\exp(\eta) \equiv \sin \phi$).

Then equation (14) becomes:

$$G = \sin \phi N(T) + T S(\lambda, \phi) + L(\lambda, \phi). \quad (14a)$$

4. Boundary conditions

The boundary conditions for the problem are

$$T = T_e(\lambda, \phi) \text{ at } z = 0, \quad (15)$$

$$w = w_e(\lambda, \phi) \text{ at } z = 0, \quad (16)$$

$$w = 0 \text{ at } z = z_b(\lambda, \phi), \quad (17)$$

where $z = 0$ represents the bottom of the surface Ekman layer, $z = z_b(\lambda, \phi)$ is the bottom surface of the thermocline, $T_e$ and $w_e$ are prescribed expressions for the nondimensionalized temperature and vertical velocity at $z = 0$. We note that at the bottom of the thermocline, $T$ tends to zero.

5. Application of boundary conditions

It is necessary to transform the solution (14a) for $G$ back into the original variables in order to apply the boundary conditions (15), (16), (17). This is done as follows. Noting that $G = p - Tz_1$ and that $\frac{\partial p}{\partial z_1} = T$ we take the partial derivative with respect to $z_1$ of equation (14a) and get

$$\frac{\partial T}{\partial z_1} [z_1 + \sin \phi N'(T) + S(\lambda, \phi)] = 0, \quad (18)$$

where $N'(T)$ denotes $dN(T)/dT$. Since $\frac{\partial T}{\partial z_1} \neq 0$ in the thermocline, then (18) gives

$$N'(T) = - \frac{[z_1 + S(\lambda, \phi)]}{\sin \phi}, \quad (19)$$

where $N$, $S$ represent arbitrary functions.
Expression (19) is rewritten as

\[ T = A(\xi) \] where \( \xi = (k z_1 / \sin \phi) + R(\lambda, \phi), \]  

(20)

where \( A, R \) represent arbitrary functions and \( k \) is an arbitrary constant. Integration of the equation

\[ \frac{\partial p}{\partial z_1} = T \]

gives

\[ p = \sin \phi \frac{B(\xi)}{k} + Q(\lambda, \phi) \]

where \( B(\xi) = \int A(\xi) d\xi \) and \( Q \) is an arbitrary function.

The function \( Q(\lambda, \phi) \) is known as the barotropic component of the pressure since it gives rise to horizontal velocity components which persist at great depth. In this paper \( Q \) is taken to be identically zero thus eliminating the barotropic components of velocity. This results in a solution which is simple enough to be amenable to detailed analysis. It is intended in a later paper to analyze the more complex situation where the barotropic velocity field is allowed to persist.

When \( p = \sin \phi \frac{B(\xi)}{k} \) then equation (4) gives

\[ u_1 = -\frac{1}{2} \sin \phi \frac{[B(\xi) + A(\xi) (-kz_1 / \sin \phi + \sin \phi R_\phi / \cos \phi)]}{k}, \]  

(21)

\[ v_1 = \frac{1}{2} \sin \phi A(\xi) R_\lambda / k. \]  

(22)

Equation (5a) gives

\[ w_1 = \frac{1}{2} \sin^2 \phi B(\xi) R_\lambda / k^2. \]  

(23)

It is easy to check that expressions (20) through (23) for \( T, p, u_1, v_1, w_1 \) satisfy equations (4), (5a), (6).

In addition to the boundary conditions (15), (16), (17) it is necessary (because of the definition of \( T \)) that \( T \to 0 \) at the bottom of the thermocline. In this paper we allow the thermocline to have infinite depth. This is an approximation to the real situation where \( z = z_b(\lambda, \phi) \) represents the bottom of the thermocline (c.f. equation (17)). For \( T \to 0 \) at the bottom of the thermocline is is necessary that \( A(\xi) \) be a decreasing function of \( \xi \). If \( A(\xi) \) is chosen so that \( B(\xi) \) is also a decreasing function of \( \xi \) (note that \( B(\xi) \) is the integral of \( A(\xi) \)) then equations (21), (22), (23) indicate that the variables \( u_1, v_1, w_1 \) all tend to zero at the bottom of the thermocline and hence boundary condition (17) is automatically satisfied. There then remains one arbitrary function \( R(\lambda, \phi) \) in the solution which can be chosen to satisfy only one of the boundary conditions (15) or (16). Two obvious decreasing functions of \( \xi \) that suggest themselves are

(a) \( A(\xi) = \exp(\xi) \)

and

(b) \( A(\xi) = (\xi)^{-\alpha} \) \( \text{where } \alpha > 0. \)
The exponential solution (a) leads to the solution derived by Welander by a different method. This is shown to be so in Appendix 1 where the details of the exponential solution are given.

The inverse power solution (b) is developed in the next section.

The exponential solution (a) and the inverse power solution (b) are two special cases of a more general set of solutions. It is intended in a later paper to analyze the properties of the general set of solutions in a model which includes barotropic velocity components and correctly accounts for a thermocline bottom of variable depth.

6. Inverse power solution

With \( A(\xi) = \xi^{-\alpha}, \alpha > 0 \), where \( \xi = k z_1 / \sin \phi + R(\lambda, \phi) \), the singularity of \( \xi^{-\alpha} \) at \( \xi = 0 \) must be avoided. Since \( z_1 \) decreases from zero, this singularity is avoided only if \( R(\lambda, \phi) \) is always negative so that \( \xi \) is always negative. It is possible to constrain \( R(\lambda, \phi) \) to be negative. \( A(\xi) \) and \( B(\xi) \) are both inverse powers of \( \xi \) if \( \alpha > 1 \) and this is the only restriction on the choice of \( \alpha \). This section develops in detail the solution for the case \( \alpha = 2 \) while Appendix 2 indicates some of the properties of the solution for \( \alpha = 3 \).

6.1 Case, \( \alpha = 2 \). Then \( A(\xi) = \xi^{-2} \) and \( B(\xi) = -\xi^{-1} \).

To satisfy boundary condition (15) (the solution can satisfy only one of condition (15) or (16)) requires

\[
R(\lambda, \phi) = \pm 1/T_e^{3/2}.
\]  

We choose the negative sign in (24) to ensure that \( \xi \) is always negative.

Then the solution for the nondimensionalized perturbation temperature, \( T \), and the nondimensionalized velocity components \( u, v, w \) is

\[
T = \xi^{-2}, \quad \xi = (-T_e^{-1/2} + k z_1 / n_1, \sin \phi),
\]

\[
u = \frac{1}{4} k \cos \phi \frac{\xi^{-2} T_e^{-3/2}}{\frac{\partial T_e}{\partial \lambda}},
\]

\[
w = -\frac{1}{4} k^2 \frac{n_1}{n_2} \frac{\xi^{-1} T_e^{-3/2}}{\frac{\partial T_e}{\partial \lambda}}.
\]

6.2 Properties of the solution. The influence of the distribution of surface temperature, \( T_e \), in the solution is clearly seen in expressions (25), (26), (27), (28). In order to illustrate the properties of the solution we assume that the surface temperature distribution is represented adequately by the functional relationship \( t_e' = \)
22 \cos (\phi + 10^\circ) \text{i.e.} \ Te = 5.5 \cos (\phi + 10^\circ) -1. \ This \ relationship \ represents \ reasonably \ closely \ the \ observed \ distribution \ of \ surface \ temperature \ (\text{c.f.} \ Needler, \ 1967) \ in \ the \ Western \ North \ Atlantic \ between \ latitude \ 20N \ and \ 60N. \ Note \ that \ the \ above \ expression \ for \ Te \ does \ not \ account \ for \ the \ longitudinal \ variation \ in \ temperature \ which \ must \ be \ present \ if \ the \ velocity \ components \ v, \ w \ are \ to \ be \ nonzero \ (\text{c.f.} \ equations \ (27), \ (28)). \ However, \ it \ is \ possible \ to \ illustrate \ certain \ properties \ of \ the \ solution \ without \ directly \ specifying \ the \ longitudinal \ variation \ of \ surface \ temperature, \ Te.

The bottom of the thermocline is defined by \( T = 0 \) which according to expression (25) occurs as \( z \to -\infty \). However, the temperature, \( t' \), is quite close to \( t' = 4^\circ \text{C} \) at finite values of \( z \). We assume that the “effective” bottom of the thermocline is defined by \( t' = 4.6^\circ \text{C} \), \text{i.e.} \( T = 0.15 \). Then the “effective” bottom of the thermocline occurs at finite values of \( z \). The solution (25) for \( T \) contains a constant, \( k \), whose value has not so far been specified. This constant can be chosen so that the “effective” thermocline bottom at one latitude satisfies a prescribed value. We choose \( k \) so that the “effective” thermocline bottom at \( \phi = 30N \) is at \( z' = 1200m \) (this value of \( z' \) accords with observations). The appropriate value of \( k \) is \( k = 0.474 \). For the surface temperature \( Te \) quoted above and \( k = 0.474 \) a north-south section of the temperature profile, \( T \), as given by expression (25) is plotted in Figure 1. As observed previously the thermocline bottom depth is largest in mid-latitudes. The decay of temperature with depth near the surface is more rapid than was observed but the decay of temperature in the lower region of the thermocline is closer to observations. It is anticipated that other choices of the constant \( k \) and the constant \( \alpha \) in the inverse power solution (here we took \( k = 0.474 \) and \( \alpha = 2 \)) would lead to a temperature profile which agreed adequately with observations. This is under investigation.

The nondimensionalized vertical velocity component, \( w \), is given by (28) where the sign of \( w \) depends on the sign of \( \partial Te/\partial \lambda \). Its value, \( We \), at the bottom of the Ekman layer is from (28)

\[
we = \frac{1}{4} \frac{k}{k^2} \left( \frac{n_1}{n_2} \right) Te^{-1} \frac{\partial Te}{\partial \lambda} .
\] (31)

This expression (31) for \( We \) is of particular interest since the boundary condition (16) for \( we \) is not satisfied by the solution; hence, it is necessary that expression (31) should represent at least the gross features of boundary condition (16) for the solution presented here to be useful. In the western Atlantic the flow beneath the Ekman layer is southward and an integration of equation (5a) implies that the vertical velocity at the bottom of the Ekman layer is downward (i.e. \( we < 0 \)). Estimates by Montgomery indicate that a typical magnitude for \( we' \) is \( we' = -5 \times 10^{-5} \text{cm/sec} \). i.e. \( we = -1 \).

The surface temperature charts given in Stommel (1965, pp. 24, 25) indicate
Figure 1. North-south section of the temperature field with surface temperature \( t_s' = 22 \cos (\phi + 10^\circ) \) where \( t_s' \) is measured in °C.
that in the western Atlantic away from the North American coast $\frac{\partial T_e}{\partial \lambda}$ is indeed negative so that (31) gives the correct sign for $w_e$. Measurements from these charts and included in Needler indicate that a typical value for $\frac{\partial T_e}{\partial \lambda}$ at 30N is $\frac{\partial T_e}{\partial \lambda} = -1$.

With this value for $\frac{\partial T_e}{\partial \lambda}$ and $T_e = 5.5 \cos (\phi + 10^\circ) - 1$ as before then expression (31) gives $w_e$ at $\phi = 30N$ to be $w_e = -0.618$ which is of the correct order of magnitude.

The velocity component, $v$, in the northward direction is given by expression (27). The value, $v_e$, at the bottom of the Ekman layer is from (27)
Here again the sign of $v_e$ and indeed of $v$ (c.f. equation (27)) depends on the sign of $\frac{\partial T_e}{\partial \lambda}$. For instance, as noted previously, $\frac{\partial T_e}{\partial \lambda}$ is negative in the western north Atlantic so that expressions (27) and (32) indicate southward flow toward
the equator. This accords with observations of a southward counterflow beneath the Ekman layer in the western north Atlantic.

The magnitude of $v'_e$ should be (c.f. Veronis, 1969; Blandford, 1965) of the order of one cm/sec i.e. $v_e$ is of order one. With the previously quoted expressions for $\frac{\partial T_e}{\partial \lambda}$ and $T_e$ expression (32) gives the value of $v_e$ at $\phi = 30^\circ$N to be $v_e = -0.34$. The sign of $v_e$ is correct but the magnitude is too small by a factor of two or three.

The velocity component, $u$, in the eastward direction is given by expression (26). The value, $u_e$, at the bottom of the Ekman layer is from (26)

$$u_e = - \frac{\cos \phi}{2k \sin \phi} T_e^{1/2} \left[ 1 + \frac{1}{2} \frac{\sin \phi}{\cos \phi} T_e^{-1} \frac{\partial T_e}{\partial \phi} \right].$$

(36)

At high latitude, where $\sin \phi / \cos \phi >> 1$, (36) gives

$$u_e \sim - \frac{1}{4k T_e^{-1/2}} \frac{\partial T_e}{\partial \phi}.$$  

(37)

Since $\partial T_e / \partial \phi < 0$, expression (37) indicates west to east surface flow at high latitudes as observed.

At low latitudes, where $\frac{\sin \phi}{\cos \phi} << 1$, (36) gives

$$u_e \sim - \frac{\cos \phi}{2k \sin \phi} T_e^{1/2} < 0.$$  

(38)

Expression (38) indicates east to west surface flow at low latitudes as observed. The magnitude of $u'_e$ should be (cf. Blandford, 1965) also of order one cm/sec i.e. $u_e$ of order one. It is possible to evaluate $u_e$ at various latitudes, $\phi$, from expression (36). With the previously quoted surface distribution, $T_e$, it is found that $u_e(\phi = 30^\circ)N = -2.235$ while $u_e(\phi = 60^\circ)N = 2.332$. These results have the correct sign and the orders of magnitude are satisfactory.

We note from expressions (25), (27), (28) that the variables $T$, $\nu$, $w$, decrease regularly downwards. The presence of the term $(1/2 + z/n_1) \xi^{-2} \cos \phi / \sin^2 \phi$ in expression (26) indicates that $u$ does not decrease regularly downwards. In fact it can be shown that expression (26) gives a minimum value for $u$ at a certain negative value of $z$. The minimum value of $u$ is $u_m$ given by

$$u_m = - \frac{1}{2k} T_e^{1/2} \frac{\cos \phi}{\sin \phi} \left[ 1 - \frac{1}{2} \frac{\sin \phi}{\cos \phi} T_e^{-1} \frac{\partial T_e}{\partial \phi} \right]^{-1},$$

(39)

and occurs at $z = z_m$ given by

$$z_m = \frac{n_1 \sin^2 \phi}{2k \cos \phi} T_e^{-3/2} \frac{\partial T_e}{\partial \phi} < 0.$$  

(40)

Also $u$ is zero if $z \to -\infty$ and if $z = z_0$ where
\[ z_0 = \frac{n_1 \sin \phi}{2k} T_e^{-1/2} \left[ 1 + \frac{1}{2} \frac{\sin \phi}{\cos \phi} \frac{T_e^{-1}}{} \frac{\partial T_e}{\partial \phi} \right]. \] (41)

The downward variations of \( u \) at two different latitudes \( \phi = 30^\circ \)N and \( \phi = 60^\circ \)N are plotted in Figure 2 and Figure 3 respectively. The same surface distribution \( T_e \) as before is used in evaluating expressions (39), (40), (41). Figure 2 indicates a westward flow which initially increases with depth and then decreases toward zero at the bottom of the thermocline. Figure 3 indicates an eastward flow which decreases regularly toward zero at the bottom of the thermocline. We note that the reverse flow indicated in Figure 3 is spurious since it occurs below the thermocline where the solution presented here is not valid.

6.3 Relation to other solutions. Fofonoff, in considering thermohaline circulation, included vertical diffusion of heat in his model. Veronis was able to produce a generalized version of the Fofonoff solution. Both these solutions are of the inverse power type. As noted by Veronis both solutions are critically dependent on \( K \), the coefficient of vertical diffusion of heat. In fact both solutions vanish entirely when \( K \) is zero and so are not related to the inverse power solution of the advective model considered here where \( K = 0 \).

APPENDIX I

The solution \( A(\xi) = \exp(\xi) \) gives \( B(\xi) = \exp(\xi) \).

Then from equation (20) \( T = A(\xi) = \exp(\xi) = \exp(kz_i/\sin \phi) \exp \left[ R(\lambda, \phi) \right] \) and the boundary condition (15) gives

\[ T = T_e (\lambda, \phi) \exp \left( kz_i/\sin \phi \right) \] (42)

where \( \exp \left[ R(\lambda, \phi) \right] = T_e (\lambda, \phi) \). Expression (42) is identical to the solution for the density field given by Welander (1959, eqn. (30)).

The nondimensionalized velocity components \( u, v, w \) are given by

\[ u = \frac{\cos \phi}{\sin^2 \phi} u_1, \quad v = \frac{1}{\sin \phi \cos \phi} v_1, \]

\[ w = \frac{(n_1/n_2)}{\sin^2 \phi} w_1 \]

together with expressions (21), (22), (23) for \( u_1, v_1, w_1 \).

Substitution of \( A(\xi) = \exp(\xi) \) gives

\[ u = \frac{e^{kz_i/(n_1 \sin \phi)}}{2k} \left[ \frac{kz_i}{(n_1 \sin \phi)} \frac{\cos \phi}{\sin \phi} T_e - \frac{\cos \phi}{\sin \phi} T_e - \frac{\partial T_e}{\partial \phi} \right], \] (43)

\[ v = \frac{1}{2k \cos \phi} \cdot \frac{\partial T_e}{\partial \lambda} \cdot \exp \left( kz_i/n_1 \sin \phi \right), \] (44)

\[ w = \frac{(n_1/n_2)}{2k^2} \frac{\partial T_e}{\partial \lambda} \exp \left( kz_i/n_1 \sin \phi \right) \] (45)

Apart from the different notation the above expressions for \( u, v, w \) correspond exactly to the solution given by Welander (1959, eqns. (31), (32)).
APPENDIX II

It is clear that if $\alpha$ is any even integer (where $A(\xi) = \xi^\alpha$) than the solution proceeds exactly as for the case $\alpha = 2$. If $\alpha$ is any odd integer ($\geq 3$) then the solution proceeds as for the case $\alpha = 3$. Some details of the $\alpha = 3$ case are outlined here.

Since $T = A(\xi)$, for $T$ positive we must take $A(\xi) = -\xi^{-3}$. To avoid the singularity of $\xi^{-3}$ at $\xi = 0$ we must ensure that $\xi = k z / n_1 \sin \phi + R(\lambda, \phi)$ is always negative as $z$ decreases from zero. For this it is required that $R(\lambda, \phi)$ is negative. The boundary condition $T = T_0(\lambda, \phi)$ at $z = 0$ requires

$$ R(\lambda, \phi) = - T_0^{-1/3}.$$

Then $R(\lambda, \phi)$ is negative and $A(\xi) = -\xi^{-3}$ where $\xi = k z / n_1 \sin \phi - T_0^{-1/3}$, and $B(\xi) = \frac{1}{2} \xi^{-2}$.

The remainder of the solution proceeds as for $\alpha = 2$.

REFERENCES


Received: 12 July, 1977; revised: 1 November, 1977.