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The propagation of wind-generated inertial oscillations from the surface into the deep ocean

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ABSTRACT

The object of this study is to explain the occurrence of relatively large amplitude inertial motion observed at great depths in the ocean. An arbitrarily stratified, $\beta$-plane model is considered consisting of a viscous boundary layer at the surface and an inviscid interior. The forcing due to a wind stress produces Ekman suction in the boundary layer which in turn drives the interior. Ray theory is then used to describe the propagation of disturbances in the boundary layer down into the interior.

It is found that the ability of energy to propagate out of the boundary layer depends on a parameter $\lambda$, the difference between the driving frequency and the local inertial frequency. In general, smaller horizontal wave numbers tend to produce larger velocity amplitudes in the deep.

Though our boundary layer analysis to obtain $\lambda$ is overly simplified, the results are consistent with gross observations of inertial motion properties. The model does provide for the existence of relatively large inertial motion in the deep ocean.

1. Introduction

Motions with nearly inertial periods are often observed in the ocean. Most measurements have been made near the surface, and the strongest inertial activity has been observed there. However, inertial motions with significant horizontal velocity amplitudes (~ 10 cm/sec) have been observed at great depths (Webster, 1968; Perkins, 1970; Pochapsky, 1966; Brekhovskikh, et al., 1971).

The major properties of the observed inertial motion can be summarized as follows:

(a) The observed frequency is generally slightly greater than the local inertial frequency (the inertial frequency which would correspond to the latitude of the observation site).

(b) The amplitude can range up to approximately 50 cm/sec near the surface, and up to 10 cm/sec in water of great depth, with values in the neighborhood of 1 cm/sec very often present.

(c) The coherence of inertial frequency motion is considerably greater in the horizontal than in the vertical direction (Webster, 1968).

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(d) The amplitude tends to correlate strongly with $N^{1/2}$ where $N$ is the Brunt-Väisälä frequency (Webster, 1969; Brekhovskikh et al., 1971).

(e) In general, there is great intermittence observed, with a clear inertial signal rarely lasting for more than a few inertial periods.

(f) Strong inertial motion near the surface has often been observed in conjunction with strong winds at the surface.

These motions have been investigated theoretically. Ekman (1905) first noted their existence in formulating the Ekman boundary layer. Rossby (1938) was not investigating inertial motion specifically, but noted its importance in the adjustment of a flow for geostrophic equilibrium. Veronis and Stommel (1956) considered a wind-driven, two layer model and determined the dispersive properties of free Rossby and inertiogravity waves. In this model, the wind stress was exerted as a body force distributed evenly throughout the top layer.

Pollard (1970) and Pollard and Millard (1970), using this type of wind-stress model, were able to describe the inertial motion reasonably well near the surface but not in the deep ocean. Blandford (1966) investigated how energy near the inertial frequency propagated. He used ray theory to show the propagation paths of individual vertical modes.

Munk and Phillips (1968) developed a free mode solution of near-inertial motions on a sphere for an arbitrary vertical stratification. They found that the horizontal velocity amplitude should vary with $N^{1/2}$, as observed, and predicted realistic vertical and horizontal coherence scales.

My objective is to investigate the occurrence of inertial motion of significant amplitude in the deep ocean, an observation that has not been adequately explained. The model differs from previous models in that the stress on the surface, generated by the wind, produces a time-dependent viscous Ekman boundary layer at the surface which in turn forces the essentially inviscid interior by means of the vertical velocity, the Ekman suction, of the boundary layer. A $\beta$-plane model with arbitrary stratification is used.

The plan of development of this paper is as follows. In Section 2 we develop the mathematical model, deriving the pertinent equations for the boundary layer and the interior. In Section 3 we develop the usual, time-dependent, Ekman-boundary-layer solution, neglecting pressure gradients, to find the Ekman suction velocity. We show that in this case the frequency of the forcing produced by the Ekman suction is exactly equal to the local inertial frequency and will not allow downward propagation into the interior. We then assume that the forcing frequency is actually a small amount $\lambda$ greater than the local inertial frequency. This does allow downward propagation. Hence in Section 4 we develop an asymptotic solution, using ray theory, describing the downward propagation within the interior.

From this solution the importance of the parameter $\lambda$ becomes even more apparent. Hence in Section 5 we derive an asymptotic solution of the boundary layer with
the inclusion of the pressure gradients which justifies our assumption of $\lambda$ and develops its functional character. In Section 6 we consider aspects pertinent to the real ocean: bottom reflection of waves, realistic stratification, intermittent forcing, and phase mixing. In Section 7 we apply the theoretical results to real ocean conditions and make a comparison with actual data.

2. The mathematical model

We make the following assumptions:

(1) The north-south scale is such that we can use the $\beta$-plane approximation ($f = f_0 + \beta y$) with the "traditional" approximation of neglecting the horizontal component of the Coriolis parameter.

(2) There is no conduction of heat $\left( \frac{dT}{dt} = 0 \right)$.

(3) Density perturbations ($\rho$) are sufficiently small to allow the use of the Boussinesq approximation and a linear equation of state, $\frac{d\rho}{dT} = \text{constant}$.

(4) We can neglect the non-linear terms. (The solution for sinusoidal forcing in time and space will actually satisfy the non-linear equation to a good approximation. This is to be expected since inertial oscillations are an exact solution of the hydrodynamic equations for an incompressible fluid with $f$ constant (O. M. Phillips, 1963).)

(5) The scale of vertical motion ($D$) is much less than that of the horizontal motion ($L$) allowing the use of hydrostatic balance in the vertical momentum equation $\left( \delta = \frac{D}{L} \ll 1 \right)$.

(6) Velocity gradients in the vertical direction dominate in the viscous terms, and the viscous ($\mu$) in the boundary layer can be approximated as a constant eddy value.

(7) The surface acts like a rigid lid for motions in the neighborhood of the inertial period, allowing a simple homogeneous boundary condition for the vertical velocity to be used at the surface.

(8) Initially there is no steady-state motion.

(9) The steady-state density stratification, $\rho_s(z)$, is depth-dependent only.

The equations of fluid motion and continuity in non-dimensional form are then

\begin{align*}
u_t - f \nu &= - P_x + E u_{zz}, \\
u_t - f \nu &= - P_y + E v_{zz}, \\
P_z &= - \rho, \\
\rho_t + B^2(z)w &= 0,
\end{align*}

and

\begin{align*}
u_x + v_y + w_z &= 0,
\end{align*}
where \( f = 1 + \beta y \). The variables are non-dimensionalized as follows:

\[
(x, y), \quad \text{the east and north coordinates, by the scale } L;
\]
\[
z, \quad \text{the upward coordinate, by } D, \text{ the depth of the ocean;}
\]
\[
(u, v), \quad \text{the eastward and northward velocity components, by } \hat{v};
\]
\[
w, \quad \text{the upward velocity, by } \delta \hat{v};
\]
\[
f, \quad \text{the Coriolis parameter, by } f_0 = 2\Omega \sin \varphi_0, \text{ where } \varphi_0 \text{ is the central}
\]
\[
\text{latitude of the } \beta\text{-plane and } \Omega \text{ is the rotation of the earth;}
\]
\[
t, \quad \text{the time, by } f_0^{-1}, \text{ since we are assuming near-inertial motion;}
\]
\[
P, \quad \text{the perturbation pressure, by } \varrho_0 L f_0 \hat{v}, \text{ where } \varrho_0 \text{ is the mean density of}
\]
\[
\text{the water column; and}
\]
\[
\varrho, \quad \text{the perturbation density, by } \varrho_0 f_0 \hat{v}/\delta g, \text{ where } g \text{ is the acceleration of}
\]
\[
g\text{ravity.}
\]

The relevant dimensionless numbers are the Ekman Number \((E)\), the aspect ratio \((\delta)\), the internal Froude Number \((F)\), and the dimensionless beta \((\beta)\). These are defined as follows:

\[
E = \frac{v}{f_0 D^2}, \quad \text{where } v = \frac{\mu}{\varrho_0};
\]
\[
\delta = \frac{D}{L};
\]
\[
F = \frac{N}{f_0}, \quad \text{where } N = \sqrt{-g \frac{\partial \varrho_0(z) \varrho_0}{\partial z}} \text{ is the Brunt-Väisälä frequency;}
\]

and

\[
\beta = \frac{L}{R} \cot \varphi_0, \quad \text{where } R \text{ is the radius of the earth.}
\]

The dimensionless quantity \(B\) is defined as \(B = \delta F\). In the ocean, \(E = 0(10^{-6})\). We identify the length scale \(L\) with the wavelength scale of the forcing in the horizontal, so if this wavelength is \(0(100 \text{ km})\) and \(D = 0(10 \text{ km})\), then \(\delta = 0(10^{-1})\) and \(\beta = 0(10^{-2})\). \(F\) varies from \(0(10^2)\) in the thermocline to \(0(1)\) near the bottom.

We assume that the system is initially at rest. For \(t > 0\) a wind stress, which for simplicity is in the eastward direction, produces a surface stress in the \(x\)-direction. We know from the solution of the so-called "spin-up problem" that an Ekman boundary layer of dimensionless thickness \(E^{1/2}\) is produced near the surface in the time of the order of one period of rotation and that the layer diffuses down quite slowly. Also we know that viscous effects are much greater in the Ekman layer than in the deeper water. So we consider two layers, a thin viscous layer over the essentially inviscid interior.

After eliminating \(\rho\) from the set of equations (2.1)–(2.5), we assume a solution of the form \((\mathbf{v},P) = (v_I,P_I) + (\hat{v},\hat{P})\), where \((v_I,P_I)\) is in the interior in which we assume
\[ \frac{\partial}{\partial z} = 0(1), \] and \((\tilde{u}, \tilde{P})\) is in the boundary layer in which we assume \(\frac{\partial}{\partial z} = 0(E^{-1/2}).\)

We then substitute this form into equations (2.1)-(2.5), separate the interior and boundary layer parts and "stretch" the \(z\) coordinate in the boundary layer letting \(\zeta = -E^{-1/2}z\). We assume \(u, \nu, w, P, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t}\) to be \(0(1)\) and \(E \ll 1\).

The boundary layer equations are

\[
\begin{align*}
\tilde{u}_t - f\tilde{v} &= -\tilde{P}_x + \tilde{u}_\zeta \zeta, \\
\tilde{v}_t + f\tilde{u} &= -\tilde{P}_y + \tilde{v}_\zeta \zeta, \\
\tilde{P}_\zeta &= B^2 E^{1/2} \tilde{w},
\end{align*}
\]

and

\[
\tilde{u}_x + \tilde{v}_y = E^{-1/2} \tilde{w}_\zeta. \tag{2.9}
\]

Dropping the subscript \(I\) and neglecting terms of order \(E\), the interior equations are

\[
\begin{align*}
\tilde{u}_t - f\tilde{v} &= -\tilde{P}_x, \tag{2.10} \\
\tilde{v}_t + f\tilde{u} &= -\tilde{P}_y, \tag{2.11} \\
\tilde{P}_z &= -B^2 \tilde{w}, \tag{2.12}
\end{align*}
\]

and

\[
\tilde{u}_x + \tilde{v}_y + \tilde{w}_z = 0. \tag{2.13}
\]

We want to find a solution for equations (2.6)-(2.9), specifically \(\tilde{w}\), which will provide a surface boundary condition for the solution to equations (2.10)-(2.13).

### 3. The Ekman boundary layer

We consider perturbations of \(0(E^{1/2})\) in equations (2.7)-(2.9). If \(\tilde{u}_x\) and \(\tilde{v}_y\) are \(0(1)\) in the boundary layer, then \(\tilde{w}_\zeta\) must be \(0(E^{1/2})\) from continuity. Since \(\partial/\partial \zeta, \partial/\partial x\) and \(\partial/\partial y\) are \(0(1)\), then \(\tilde{v}\) and \(\tilde{u}\) are \(0(1)\) and \(\tilde{w}\) is \(0(E^{1/2})\). Thus, if \(B^2\) is \(0(1)\), \(\tilde{P}_\zeta\) is \(0(E)\). Hence, \(\tilde{P}_y\) and \(\tilde{P}_x\) are \(0(E)\). So we neglect the pressure through \(0(E^{1/2})\) and, letting \(\psi = \tilde{u} + i\tilde{v}\), derive the equation

\[
\psi_t + if\psi = \psi_{\zeta\zeta}, \tag{3.1}
\]

which is valid for \(0(1)\) and \(0(E^{1/2})\).

The stress on the surface provides the top boundary condition. Since \(\tilde{v}_z = 0(1)\) and \(\tilde{v}_z = 0(E^{-1/2})\) in the boundary layer, we neglect the interior contribution to the stress. Then the dimensional stress in the \(x\) direction is given by

\[
\tau_d^{(x)} \propto -\frac{\hat{v}}{D} \varrho_0 \nu E^{-1/2} \tilde{u}_\zeta (\zeta = 0). \tag{3.2}
\]

(We assume \(\tau_d^{(y)} = 0\) for simplicity.)
\( \psi \) is a boundary value quantity and so must vanish as \( \zeta \to \infty \). The initial condition is that there be no motion for \( t < 0 \). We assume a stress which is impulsive in time in the form \( \tau_d(x) = \tau_0(x,y) \delta(t) \). By means of a Laplace transform, we solve equation (3.1) to obtain

\[
\psi = \frac{Y(x,y)}{\sqrt{\pi Et}} e^{-\frac{\xi^2}{4t}} e^{i\xi} U(t) \tag{3.3}
\]

where \( Y(x,y) = \frac{\tau_0(x,y)E^{1/2}}{v_0 \sqrt{fo}} = \frac{\tau_0(x,y)}{v_0 Df_o} \) and \( U(t) \) is the unit step function. This result is consistent with that originally derived by Ekman (1905).

From continuity we find \( \bar{w} \):

\[
\bar{w} = E^{1/2} \text{Im} \int \left( \psi_y + i\psi_x \right) d\zeta' \tag{3.4}
\]

\[
= \text{Erfc} \left( \frac{\zeta}{\sqrt{4t}} \right) \text{Im} \left\{ (\partial_y + i\partial_x) (Y(x,y)e^{i\xi y}) \right\} U(t).
\]

Since we assume that the total vertical velocity (the interior plus the boundary layer part) vanishes at the surface, the Ekman suction is given by

\[
\bar{w}^0 = -\bar{w}(\zeta = 0) = -\text{Im} \left\{ (\partial_y + i\partial_x) (Y(x,y)e^{i\xi y}) \right\} U(t). \tag{3.5}
\]

We do the indicated differentiation and take the imaginary part. We then assume that \( Y(x,y) \) is sinusoidal in \( x \) and \( y \) by letting \( \tau_0(x,y) = \tau_0 e^{-i(\beta y + kx)} \), where \( \tau_0 \) is constant, \( k = \frac{2\pi L}{L_x} \) and \( l = \frac{2\pi L}{L_y} \) with \( L_x \) and \( L_y \) being the wavelengths in the indicated directions. Expressing the resulting sine and cosine terms in exponential form, the result is

\[
\bar{w}^0 = K_i [k + i(\beta t - l)] e^{i(\beta t - kx - ly)} U(t) + K_i [k + i(\beta t + l)] e^{-i(\beta t + kx + ly)} U(t), \tag{3.6}
\]

where \( K = \frac{\tau_0 E^{1/2}}{2v_0 \sqrt{fo}} = \frac{\tau_0}{2v_0 Df_0} \).

From equation (3.6) we see that because of the variation of the Coriolis parameter with latitude, we have Ekman suction even if the wind stress has no curl. We might visualize this in the following manner. If we consider a uniform stress over a horizontal plane, fluid particles in the neighborhood of a given latitude move approximately in a circle on the plane. The period for a particle to make one cycle is the inertial period \( \frac{2\pi}{f} \), which depends on latitude. We therefore have water particles at different latitudes moving in circles out of phase with each other. Such motion will tend to "pile" or "trough" the water, making some kind of vertical motion necessary to preserve continuity.
To this order of approximation for the Ekman layer, the amplitude grows linearly with time. However, when the pressure gradient is included at \( O(E) \), there is eventual exponential decay of the Ekman suction velocity. This is shown in detail in Section 5.

Equation (3.6) dynamically connects the boundary layer with the interior. It is the driving force at \( z = 0 \) for the interior. However, it is important to note that the driving frequency is equal to \( f \). That is, the driving frequency at a particular latitude designated by \( \beta \) is equal to exactly the value of the inertial frequency at that latitude.

If we were to assume \( f \) and \( B \) to be constant in the interior equations (2.10)-(2.13) and assume a plane wave solution of the form \( e^{i(\omega t - kx - \beta y - mz)} \), the resulting dispersion relation would be \( \omega = \sqrt{f^2 + \frac{B^2}{m^2}} (k^2 + l^2) \). For wave propagation, \( \omega \) must exceed \( f \), implying that our forcing at exactly \( f \) should not propagate.

If the inertial motion is being propagated down into the interior from the boundary layer, then the forcing frequency must be larger than \( f \). We therefore assume that this frequency is a small amount, \( \lambda \), greater than the local inertial frequency corresponding to a particular latitude. We justify this assumption in Section 5 for the linear boundary layer where we include the pressure gradients which come into equations (2.6)-(2.9) at \( O(E) \). Neglecting derivatives in \( x \), we find that \( \lambda (\ll f) \) is a function of the horizontal wave numbers and time.

Hence, we modify the Ekman suction forcing equation (3.6) to be

\[
\omega^0 = iK [k + i(\beta t - l)] e^{i(\omega t - kx - \beta y)} U(t), \quad (3.7)
\]

where \( \omega = f + \lambda \). We assume that the wave numbers \( k \) and \( l \) can be positive or negative, so that we need not include the second term of equation (3.6) which can be derived from the first term letting \( l = -l, k = -k \), and taking the conjugate.

4. The inviscid interior

Now let us consider the interior equations (2.10)-(2.13). It is convenient to transform the vertical coordinate such that \( \xi = \frac{1}{\hat{B}} \int_{0}^{z} B(z) \, dz \) (or equivalently \( \xi = \frac{1}{\hat{F}} \int_{0}^{z} F(z) \, dz \) where \( \hat{B} = \int_{0}^{1} B(z) \, dz \) and \( \hat{F} = \int_{0}^{1} F(z) \, dz \). Here we let \( z \) and \( \omega \) be positive downward, and we call \( \xi \) the transformed depth. (This transformation would come naturally if the method that follows were applied directly to equations (2.10)-(2.13).)

The new set of equations with the pressure eliminated is

\[
u_{\xi t} - f v_{\xi} = B(\xi) \hat{B}w_x, \quad (4.1)
\]

\[
u_{\xi t} + f u_{\xi} = B(\xi) \hat{B}w_y, \quad (4.2)
\]

and

\[
u_y + u_x + \frac{B(\xi)}{\hat{B}} w_{\xi} = 0. \quad (4.3)
\]
If we eliminate \( u \) between equations (4.1) and (4.2), we have

\[
\frac{1}{B(\xi)} (v_{\xi\xi\xi} + f^2 v_{\xi}) = \dot{B}(w_{\eta} - f w_x). \tag{4.4}
\]

Now, eliminating \( w \) between equations (4.1) and (4.2) and using the continuity equation (4.3) to eliminate \( w \) in equation (4.4), we have

\[
B(\xi) \left[ \frac{1}{B(\xi)} (v_{\xi\xi\xi} + f^2 v_{\xi}) \right]_{\xi} + \dot{B}^2 (v_{xy} - f v_x) = -\dot{B}^2 \partial_x (u_{yt} - f u_x) \tag{4.5}
\]

and

\[
v_{xt} + f v_y + \beta v = (u_{yt} - f u_x). \tag{4.6}
\]

Then eliminating the quantity \( (u_{yt} - f u_x) \), we have the equation to be solved:

\[
\partial_t \{ v_{\xi\xi\xi} + f^2 v_{\xi} - g(\xi) (v_{\xi\xi} + f^2 v_{\xi}) + \dot{B}^2 (v_{xx} + v_{yy}) \} + \beta \dot{B}^2 v_x = 0 \tag{4.7}
\]

where \( g(\xi) = \frac{B'(\xi)}{B(\xi)}. \)

We are looking for solutions for times significantly larger than \( t = 0(1) \) which corresponds to a dimensional time period of the order of the inertial period divided by \( 2\pi \). We define a new time scale, \( \sigma \geq 0 \), such that

\[
t = i \sigma, \tag{4.8}
\]

where \( i \) is assumed large and \( \sigma = 0(1) \). Then we have

\[
\partial_{\sigma} \{ v_{\xi\xi\xi\sigma} + i^2 f^2 v_{\xi\xi} - g(\xi) (v_{\xi\xi\sigma} + i^2 f^2 v_{\xi}) + \dot{B}^2 i^2 (v_{xx} + v_{yy}) \} + \beta i^3 \dot{B}^2 v_x = 0. \tag{4.9}
\]

We use ray theory (Keller, 1958; Sechler and Keller, 1959; Buchal and Keller, 1960) to find an asymptotic solution to equation (4.9). Initially, we assume a solution in the form

\[
v = V(x, y, \xi, \sigma) e^{iS(x, y, \xi, \sigma)}, \tag{4.10}
\]

where \( V = V_0 + \frac{1}{i} V_1 + \ldots \). \( S \) is the eiconal part and \( V \) the amplitude part of (4.10).

We substitute this form into equation (4.9) and order in powers of \((1/i)\). We assume \( \beta = 0(1/i) \) which will make the explicit \( \beta \) term in equation (4.9) of second order, but we do not order the \( \beta \) terms contained in \( f(y) \) because the variation of \( f \) can easily be dealt with in the zeroth order.

**a. The eiconal equation.** For the zeroth order, we have the eiconal equation

\[
F(x, y, \xi, \sigma, S_x, S_y, S_{\xi}, S_{\eta}) = S_{\sigma} S_\xi S_{\xi} - f^2(y) S_{\xi}^2 - \dot{B}^2 (S_x^2 + S_y^2) = 0. \tag{4.11}
\]

This equation can be solved by the general method of solving non-linear, first-order partial differential equations (Sneddon, 1957). Defining \( p = S_{\xi} \), \( q = S_y \), \( r = S_x \) and
The characteristics, which are also called rays, of equation (4.11) are found by solving the following set of ordinary differential equations:

$$\frac{d\xi}{F_p} = \frac{dy}{F_q} = \frac{dx}{F_r} = \frac{d\sigma}{F_H} = \frac{dp}{-F_x} = \frac{dr}{-F_y} = \frac{dq}{-F_y} = \frac{dH}{-F_H},$$

(4.12)

along with the equation (4.11).

We then find that

$$p = \text{constant on a ray},$$

(4.13a)

$$r = \text{constant on a ray},$$

(4.13b)

$$H = \text{constant on a ray},$$

(4.13c)

$$q = \pm \left(\frac{p^2}{B^2}(H^2 - f^2(y)) - r^2\right),$$

(4.13d)

$$\frac{dx}{d\sigma} = -\dot{B}^2 r,$$

(4.13e)

$$\frac{dy}{d\sigma} = -\dot{B}^2 q,$$

(4.13f)

and

$$\frac{d\xi}{d\sigma} = \frac{\dot{B}^2}{p^2 H} (q^2 + r^2).$$

(4.13g)

In terms of the time scale $\sigma$, the quantities $r, q,$ and $p$ can be identified as wave numbers, the quantity $H$ as a frequency, and equations (4.13e, f, and g) as group velocity components. The group velocity is the velocity of propagation of a disturbance from the surface into the interior. It can also be thought of as the velocity of a wave packet of energy. A ray is the path in space generated by the trajectory of the propagation, being tangential to the group velocity vector at every point. So we say that a disturbance travels down a ray.

At the surface ($\xi = 0$), the solution to the eiconal equation (4.11) must match the eiconal part of the Ekman forcing, equation (3.7). Expressing the eiconal part of equation (3.7) in terms of $\sigma$, we find at $\xi = 0$,

$$S = S^0 = \omega \sigma - \ell' y - k' x,$$

(4.14a)

$$r = S_x = -k',$$

(4.14b)

$$q = q_0 = \beta \sigma - \ell',$$

(4.14c)

and

$$H = S_\sigma = \omega,$$

(4.14d)

where $k' = k/\ell$ and $\ell' = l/\ell$. If a given ray emanates from a particular point on the surface, $(x_0, y_0, 0)$ at a particular time $\sigma = \sigma_0$, then these conditions at $\xi = 0$ become

$$S = S^0 = \omega_0 \sigma_0 - \ell' y_0 - k' x_0,$$

(4.15a)
\begin{equation}
\begin{aligned}
    r &= -k', \\
    q &= q_0 - \beta \sigma_0 - l', \\
    H &= \omega_0 = f_0 + \lambda,
\end{aligned}
\end{equation}

and

where \( f_0 = 1 + \beta \gamma_0 \).

Using the condition (4.15c), we find that the proper sign of \( q \) in (4.13d) is determined by the sign of \( q_0 \). Hence

\begin{equation}
    q = s \sqrt{\frac{p^2}{B_2^2 (\omega_0^2 - f^2(y))} - k'^2}
\end{equation}

where \( s = \begin{cases} -1 & q_0 < 0 \\ 1 & q_0 > 0 \end{cases} \). By evaluating (4.16) at the surface, we find

\begin{equation}
    p = \frac{B \sqrt{q_0^2 + k'^2}}{\sqrt{\omega_0^2 - f_0^2}}.
\end{equation}

The sign of \( p \) is determined by the radiation condition which in this case demands that energy flow downward away from the boundary layer. This implies that the vertical component of the group velocity (4.13g) evaluated at the surface be positive. Hence, \( p \) must be positive since \( H = \omega_0 \) is always positive.

Consider the components of the group velocity given by (4.13e, f, and g). The direction of the vertical (\( \xi \)) component is always positive unless there is bottom reflection. The direction of the \( x \)-component is positive for \( k' > 0 (r < 0) \) and negative for \( k' < 0 (r > 0) \). The direction of the \( y \)-component depends on the sign of \( q \). At the surface it is positive for \( q_0 < 0 \) and negative for \( q_0 > 0 \). Away from the surface, we must take into account that \( q \) varies with \( f(y) \). Using the equation (4.17) for \( p \) in \( q \) from (4.17), we find that

\begin{equation}
    q = \frac{s \sqrt{q_0^2 + k'^2}}{\sqrt{\omega_0^2 - f_0^2}} \sqrt{f_m^2 - f^2(y)},
\end{equation}

where

\[ f_m^2 = \frac{q_0^2 \omega_0^2 + k'^2 f_0^2}{q_0^2 + k'^2} = f_0^2 + \frac{(2f_0 \lambda + \lambda^2) q_0^2}{q_0^2 + k'^2}. \]

For \( f > f_m \), \( q \) becomes imaginary.

So the group velocity is real only for \( f < f_m \).

We now visualize a disturbance generated at the point on the surface \((x_0, y_0, 0)\) at the time \( \sigma_0 \). If \( q_0 = \beta \sigma_0 - l > 0 \), the \( y \)-component of the group velocity is negative (southward) at \( y = y_0 \) and remains so for decreasing \( y \). This is because \( f(y) \) decreases with decreasing \( y \) and will always remain less than \( f_m \). If \( q_0 < 0 \), the \( y \)-component of the group velocity is positive (northward) at \( y = y_0 \) but will eventually vanish as \( y \) increases to the point where \( f(y) = f_m \). The disturbance cannot go any further north than this, and the sign of \( s \) must change to positive so that the disturbance can subsequently turn back to travel southward.

The rays can be found by integrating the group velocity components. We use the
boundary condition that at $\sigma = \sigma_0$, we have $x = x_0$, $y = y_0 (f = f_0)$ and $\xi = 0$. The rays are then given by the following parametric set of equations with $\sigma$ the independent parameter:

$$x - x_0 = \frac{B^2 k'}{p^2 \omega_0} (\sigma - \sigma_0), \quad (4.19a)$$

$$f - f_0 = -f_0 \left[ 1 - \cos \left( \frac{B \beta (\sigma - \sigma_0)}{p \omega_0} \right) \right] \sqrt{f_m^2 - f_0^2} \sin \left( \frac{s \beta (\sigma - \sigma_0)}{p \omega_0} \right), \quad (4.19b)$$

and

$$\xi = \frac{B \beta^2}{\omega_0 p^3} (\sigma - \sigma_0) + \frac{f_m^2}{2 \beta B} \frac{\dot{B} \beta}{\omega_0 f_m} (\sigma - \sigma_0) + \frac{s}{2} \left( \cos 2b - \sin 2b_0 \right), \quad (4.19c)$$

where

$$b = \frac{s \dot{B} \beta}{p \omega_0} (\sigma - \sigma_0) + b_0 \quad \text{and} \quad b_0 = -\sin^{-1} \frac{f_0}{f_m}.$$ 

The quantity $\epsilon = \frac{\dot{B} \beta (\sigma - \sigma_0)}{p \omega_0}$ can be shown to be small under realistic conditions.

Using equation (4.17) for $p$, letting $\sigma = \dot{t}^{-1} t$, $\sigma_0 = \dot{t}^{-1} t_0$, $k' = \dot{t}^{-1} k$ and $l' = \dot{t}^{-1} l$, we obtain

$$\epsilon = \frac{\sqrt{2f_0 \lambda + \beta^2 (t - t_0)}}{\omega_0 \sqrt{k^2 + (\beta t - l)^2}}. \quad (4.20)$$

For $L = L_{x} = L_y = 0(10^2 \text{km})$, we have $k = l = 2\pi$ and $\beta = 0(10^{-2})$. If $\lambda = 0(10^{-1})$ and $(t - t_0) = 0(10^3)$, then $\epsilon$ is $0(10^{-1})$. Hence we assume $\epsilon$ is sufficiently small so that $\sin \epsilon \approx \epsilon$ and $\cos \epsilon \approx 1 - \epsilon^2/2$. Then the rays become

$$x - x_0 = \frac{B^2 k'}{p^2 \omega_0} (\sigma - \sigma_0), \quad (4.21a)$$

$$y - y_0 \approx -\frac{\dot{B}^2 f_0}{2 \omega_0 p^2} \left[ (\sigma - \bar{\sigma})^2 - (\sigma_0 - \bar{\sigma})^2 \right] + \frac{2\lambda}{f_0} (\sigma - \sigma_0) (\sigma_0 - \bar{\sigma}), \quad (4.21b)$$

$$\xi \approx \frac{k' B^2}{\omega_0 p^3} (\sigma - \sigma_0) + \frac{\dot{B}^2 f_0}{3 \omega_0 p^3} \left[ (\sigma - \bar{\sigma})^3 - (\sigma_0 - \bar{\sigma})^3 \right] + \frac{2\lambda}{f_0} \left[ 3 (\sigma_0 - \bar{\sigma})^2 (\sigma - \sigma_0) + \frac{3}{2} (\sigma_0 - \bar{\sigma}) (\sigma - \sigma_0)^2 - \frac{\beta^2 (\sigma_0 - \bar{\sigma})^2}{(k'^2 + q_0^2)} (\sigma - \sigma_0)^3 \right], \quad (4.21c)$$

where $\bar{\sigma} = l'/\beta$ and $q_0 = \beta (\sigma_0 - \bar{\sigma})$.

In general, we may look upon the ray equations as representing a coordinate transformation from the field variables, $x, y, \xi,$ and $\sigma$, to a new set of coordinates, $x_0, y_0, \sigma_0,$ and $\sigma$, which we call the ray variables. The characteristics of the eiconal equation provide the natural transformation to best describe wave propagation and facilitate a solution to the problem.
The phase function, \( S \), can be calculated on a ray using the equation

\[
\frac{dS}{d\sigma} = r \frac{dx}{d\sigma} + q \frac{dy}{d\sigma} + p \frac{d\xi}{d\sigma} + H.
\]

All of the quantities in this equation have been calculated on a ray (4.13). Upon substitution, we find,

\[
\frac{dS}{d\sigma} = \omega_0,
\]

which can be integrated to yield

\[
S(x_0, y_0, \sigma_0, \sigma) = \omega_0\sigma - l'y_0 - k'x_0.
\]

The constant of integration was evaluated using the condition (4.15a) for \( \sigma = \sigma_0 \).

**b. The amplitude equation.** The next step is to calculate the amplitude, \( V_0 \). We now go to the first order of the expansion in \((1/i)\) generated by the substitution of (4.10) into (4.9). The resulting equation can be transformed to an ordinary first order differential equation on a ray by using the relation

\[
\frac{dV_0}{d\sigma} = V_{0x} \frac{dx}{d\sigma} + V_{0y} \frac{dy}{d\sigma} + V_{0\xi} \frac{d\xi}{d\sigma} + V_{0\sigma}.
\]

After some manipulation, we obtain the amplitude equation.

\[
-2Hp^2 \frac{dV_0}{d\sigma} + V_0 \left\{ \tilde{B}^2(q_y + r_x) - H_\sigma p^2 - p_\xi (H^2 - f^2) - 2H(p^2)\sigma + g(\xi)p(H^2 - f^2) \right\} = 0.
\]

The partial derivatives in (4.23) are in terms of the field variables \((x, y, \xi, \sigma)\). But to solve (4.23) along a ray, these partials must be described in terms of the ray variables \((x_0, y_0, \sigma_0, \sigma)\). The Jacobian of the transformation is

\[
J \begin{pmatrix} x, y, \xi, \sigma \\ x_0, y_0, \sigma_0, \sigma \end{pmatrix} = \begin{vmatrix} x_{x_0} & y_{x_0} & \xi_{x_0} & \sigma_{x_0} \\ x_{y_0} & y_{y_0} & \xi_{y_0} & \sigma_{y_0} \\ x_{\sigma_0} & y_{\sigma_0} & \xi_{\sigma_0} & \sigma_{\sigma_0} \\ x_\sigma & y_\sigma & \xi_\sigma & \sigma_\sigma \end{vmatrix}.
\]

If \( \varphi \) is any dependent variable, then the derivative with respect to \( y \), for example, is given by
\[
\Phi_y = \frac{J \left( \frac{x, \varphi, \xi, \sigma}{x_0, y_0, \sigma_0, \sigma} \right)}{J \left( \frac{x, y, \xi, \sigma}{x_0, y_0, \sigma_0, \sigma} \right)}
\]

(4.25)

in terms of the ray variables.

The variables \(x, y, \) and \(\xi\) are known as functions of the ray variables from (4.21a, b, c), and the quantities \(r, q, p,\) and \(H\) are known from (4.15b), (4.18), (4.17), and (4.15d) respectively. Then the quantity \([\hat{B}^2(q_y + r_x) - H_\sigma \sigma^2 - p_\xi (H^2 - f^2) - 2H(p^2)_\sigma]\) in (4.23) which we define as \(Q\) becomes

\[
Q = \frac{1}{J} \left\{ \hat{B}^2(q_{y0} \xi_{o\sigma} - \xi_{y0} q_{o\sigma}) - p^2 \beta(y_{o\sigma} \xi_{\sigma} - \xi_{o\sigma} y_{\sigma}) - \frac{\hat{B}^2}{p^2} (q^2 + r^2)(y_{y0} p_{o\sigma} - p_{y0} y_{o\sigma}) - 4\omega_0 p [p_{y0} (y_{o\sigma} \xi_{\sigma} - \xi_{o\sigma} y_{\sigma}) - p_{o\sigma} (y_{y0} \xi_{\sigma} - \xi_{y0} y_{\sigma})] \right\}
\]

(4.26)

where \(J\) is Jacobian of the transformation from (4.24).

We are assuming \(\lambda \ll f_0\) and \(\beta = 0 \left( \frac{1}{\lambda} \right)\), so we may simplify the evaluation of \(Q\) in (4.26) by neglecting terms small in \(\lambda\) and \(\lambda^{-1}\). For instance, we neglect the \(\lambda\) terms in (4.21b, c). But we do not neglect \(\lambda\) in \(\omega_0 = f_0 + \lambda\) in the eiconal (4.22) because a small frequency difference is crucial, as we have shown. Remembering that \(\sigma = 0(1), \hat{B} = 0(1), k' = 0(1/\lambda),\) and \(l' = 0(1/\lambda)\) which implies \(q_0 = 0(1/\lambda)\), we find \(p = 0(\lambda^{-1/2})\) \(0(1/\lambda)\) from (4.17). Then evaluating all the derivatives in (4.26), we find \(p_{y0} = 0(\lambda^{-1/2}) 0(1/\lambda), p \sim p_{o\sigma} = 0(1/\lambda) 0(\lambda^{-1/2}); q \sim q_\sigma \sim q_{o\sigma} = 0(1/\lambda); q_{y0} = 0(1/\lambda); x_\sigma \sim x_{o\sigma} \sim y_\sigma \sim y_{o\sigma} = 0(1/\lambda); \xi_\sigma \sim \xi_{o\sigma} = 0(1/\lambda) 0(1/\lambda); \xi_{y0} = 0(1/\lambda) 0(1/\lambda); x_{y0} \sim y_{y0} = 0(1/\lambda); y_{x0} = \xi_{x0} = \sigma_{x0} = \sigma_{y0} = \sigma_{o\sigma} = 0;\) and \(\sigma_\sigma = 1.\) Ordering in \(\lambda\) and \((1/\lambda)\), we find that (4.26) becomes

\[
Q \approx \frac{\partial p}{\partial \sigma_0} \left\{ 4\omega_0 p \frac{\partial \xi}{\partial \sigma} \frac{\partial y}{\partial y_0} - \frac{\hat{B}^2}{p^2} (q^2 + k^2) \frac{\partial y}{\partial y_0} \right\} \left[ 1 + 0(\lambda) + 0 \left( \frac{1}{\lambda} \right) + 0(\lambda) 0 \left( \frac{1}{\lambda} \right) + \ldots \right].
\]

(4.27)

Noting that from (4.21c)
\[
\frac{\partial \xi (\sigma_0, p(\sigma_0))}{\partial \sigma_0} = \frac{\partial \xi}{\partial \sigma_0} + \frac{\partial \xi}{\partial p} \frac{\partial p}{\partial \sigma_0},
\]

we can evaluate the derivatives in (4.27) to obtain to lowest order in \(\lambda\) and \(\lambda^{-1}\)
where $G(\sigma) = k'' + 2k'\beta^2(\sigma_0 - \bar{\sigma})^2 + 3\beta^2(\sigma_0 - \bar{\sigma}) \left[ k'(\sigma - \sigma_0) + \frac{\beta^2}{3}(\sigma - \bar{\sigma})^3 \right]$. We note that $B_1; \xi = -\frac{dB}{d\xi}$, where $\frac{dB}{d\sigma}$ is known from (4.13g). Noting also from the eiconal equation (4.11) that $(H^2 - f(y)^2) = \frac{\beta^2}{p^2}(r^2 + q^2)$, the term $g(\xi)p(H^2 - f^2)$ in (4.23) becomes $\frac{p^2f_0 dB}{B \frac{dB}{d\sigma}}$. We can now solve (4.23) easily to obtain

$$V_0 = C_0 \left( \frac{B(\xi(\sigma))}{G(\sigma)} \right)^{1/2},$$

(4.29)

where $C_0$ is the arbitrary constant of integration.

To evaluate $C_0$, we must again consider the top boundary condition (3.7) and now utilize the amplitude part. Since this boundary condition is in terms of $w$, we go back to (4.4) to find $V_0$ in terms of $W_0$ where we consider an expansion in $w$ similar to that for $v$ in (4.10). To lowest order in $(1/i)$, we find

$$W_0 = -\frac{pH(H^2 - f^2(y))}{BB(\xi)(qH + ifr)} V_0.$$  

At the surface where $\xi = 0$, $\sigma = \sigma_0$, $x = x_0$, and $y = y_0$, we find

$$W_0 = -\frac{\hat{B}(q_0 + ik'')}{pB(\xi = 0)} V_0,$$

(4.30)

where we have again used $(H^2 - f^2) = \frac{\beta^2}{p^2}(r^2 + q^2)$ and neglected terms of $O(\lambda)$. From (3.7) in terms of $\sigma$, we have

$$W_0 = -iK [q_0 - ik'] U(\sigma_0)$$

(4.31)

at the surface. Hence using (4.31) in (4.30), we can find $V_0$ at the surface and evaluate $C_0$ from (4.29).

We then have the following expression for the zeroth order northward velocity amplitude in terms of the ray variables:

$$V_0 = \frac{iK \sqrt{B_0 B} \sqrt{q_0^2 + k'^2(q_0 - ik')^2}}{\sqrt{2f_0 \lambda \sqrt{G(\sigma)}}} U(\sigma_0),$$

(4.32)

where $K = \frac{\tau_0}{2Vq_0Df_0}$, $q_0 = \beta\sigma_0 - l'$, $B_0 = \delta F_0 = B(\xi = 0)$, and $G(\sigma)$ is given by (4.28).

We should note here that we can assume that any of the parameters associated with the forcing, like $K$ or $\lambda$, could be slowly varying functions of $x$, $y$, and $\sigma$. Slow is
defined in the sense that, for any parameter $\varphi$ and any independent variable $x$, $\varphi(x)/\varphi$ must be much less than $S_{\alpha}$. Then a slowly varying parameter $\varphi(x,y,\sigma)$ can be dealt with by assuming $\varphi = \varphi(x_0,y_0,\sigma_0)$.

c. Discussion of the ray solution. At this point, let us summarize our results to the lowest order in $\lambda$ and $i^{-1}$, expressing the equations in terms of $t = \sigma \tilde{t}$, $t_0 = \sigma_0 \tilde{t}$, $k = k' \tilde{t}$, and $l = l' \tilde{t}$. The rays are given by

$$
\begin{align*}
    x - x_0 &= \frac{\hat{B}^2 k}{p^2 f_0} \left( t - t_0 \right) \left( \frac{dx}{dt} = \frac{\hat{B}k}{p^2 f_0} \right), \\
    y - y_0 &= -\frac{\beta \hat{B}^2}{2 f_0 \rho^2} \left[ \left( t - \tilde{t} \right)^2 - \left( t_0 - \tilde{t} \right)^2 \right] \left( \frac{dy}{dt} = \frac{-\beta \hat{B}^2}{f_0 \rho^2} \left( t - \tilde{t} \right) \right), \\
    \xi &= \frac{\beta^2 \hat{B}^2}{3 f_0 \rho^3} \left[ \left( t - \tilde{t} \right)^3 - \left( t_0 - \tilde{t} \right)^3 \right] + \frac{k^2 \hat{B}^2}{f_0 \rho^3} \left( t - t_0 \right) \left( \frac{d\xi}{dt} = \frac{\beta^2 \hat{B}^2}{f_0 \rho^3} \left( t - \tilde{t} \right)^2 + \frac{k^2 \hat{B}^2}{f_0 \rho^3} \right). 
\end{align*}
$$

The amplitude is given by

$$
V_0 = \frac{K \sqrt{F_0 F(\xi)} (\tilde{q}_0^2 + k^2)^{1/2} (\tilde{q}_0 - ik)^2 U(t_0)}{\sqrt{2 f_0 \lambda} \left[ k^4 - k^2 \tilde{q}_0^2 + \tilde{q}_0 (3k^2 q + \tilde{q}^3) \right]^{1/2}}. 
$$

The eiconal is given by

$$
\hat{S}(x_0,y_0,t_0,t) = \hat{t} S(x_0,y_0,\sigma_0,\sigma) = \omega_0 t - l y_0 - k x_0. 
$$

The turning latitude (from (4.18)) is given by

$$
f_m = \left[ f_0^2 + \frac{2 f_0 \lambda \tilde{q}_0^2}{\tilde{q}_0^2 + k^2} \right]^{1/2}. 
$$

The defined quantities are

$$
\begin{align*}
    \tilde{q} &= \beta (t - \tilde{t}), \\
    \tilde{q}_0 &= \beta (t_0 - \tilde{t}), \\
    \tilde{p} &= \frac{\hat{B} \sqrt{k^2 + \tilde{q}_0^2}}{\sqrt{2 f_0 \lambda}}, \\
    K &= \frac{\tau_0}{2 V_0 f_0 L}, \\
    \omega_0 &= f_0 + \lambda, \\
    f_0 &= 1 + \beta \gamma_0, 
\end{align*}
$$

where $\tilde{t} = l/\beta$ and $F_0 = F(\xi = 0)$. 

If we go to the next order in $\tilde{\tau}^{-1}$, we could find $V_1$, but the algebraic complexity would be formidable. However, we want to know the error involved in neglecting $V_1$ and higher orders. $V_1$ was found for the special case where $k = 0$ (Kroll, 1973). In this case it was found that both the quantities $(t_0 - \tilde{\tau})^2/\lambda (t - \tilde{\tau})^3$ and $\frac{\omega_0 \sqrt{2 \lambda}}{B \beta (t_0 - \tilde{\tau})}$ \(\int_0^\xi \left(\frac{B \xi'}{B}\right)^2 d\xi'\) must be small to be able to neglect $V_1$ and higher orders. Assuming that $k$ comes into these relations analogous to its appearance in the amplitude part of (3.7), an estimate for $k \neq 0$ would be that both $\left|\frac{\beta [k + i(\beta t_0 - \tilde{\tau})]}{\lambda [k + i(\beta t - \tilde{\tau})]^3}\right|$ and $\left|\frac{\omega_0 \sqrt{2 \lambda}}{B [k + (\beta t_0 - \tilde{\tau})]} \int_0^\xi \left(\frac{B \xi'}{B}\right)^2 d\xi'\right|$ must be small. Except for special cases, these criteria should be met under realistic conditions. Clearly there are problems as $\lambda \to 0$, as well as for the case where the radical in the denominator of (4.34) approaches zero. For the former case, a calculation was made for $k = 0$ (Kroll, 1973), and it was found that the field in the interior was weak with most of the energy remaining in the boundary layer. The latter case will be discussed later.

In general, these criteria become better fulfilled as $t$ increases. But we have already made the assumption that the quantity $\epsilon$ in (4.20) is small, which means that the time cannot be too large either. The spread of upper and lower limits on $t$ depends on $l$ and $k$. For wave numbers corresponding to wavelengths of the order of 100 km, the spread at mid-latitudes is $10^2 < t < 3 \times 10^6$ which corresponds to a period greater than a day and less than a month.

To recapitulate, we have found an asymptotic ray solution to lowest order which is interpreted as follows. A disturbance is generated at the surface, $\xi = 0$, at a particular location $x = x_0$, $y = y_0$, at a particular time $t = t_0$. The rays given by (4.33) describe the path which this particular disturbance will follow in the fluid as $t$ increases from $t_0$. Equations (4.34) and (4.35) describe how the amplitude ($V_0$) and the eiconal ($S$), respectively, change as the disturbance travels down a ray. As previously stated, this may be thought of as a wave packet of energy traveling down a ray with the group velocity $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{d\xi}{dt}\right)$.

The ray equations (4.33) represent a family of curves in the space $(x, y, \xi)$ defined by the ray variables $x_0, y_0, t_0,$ and $t$, the wave numbers $k$ and $l$, and the parameter $\lambda$. Since $k$ and $l$ can each be positive or negative, there are four possible combinations of $(k, l)$ for given magnitudes of these parameters. If we choose particular values for $x_0, y_0, t_0,$ and $\lambda$, we can then plot this set of four curves for each of the four combinations. A typical set of rays from one point is shown in Figure 1.

In this figure we assume the magnitude, $|l|$, of the wave number in $y$ is such that $\tilde{q}_0 = \beta t_0 - l$ is less than zero when $l$ is positive. This is to make possible the existence of the rays which go north from $(x_0, y_0)$ since, as explained earlier, $\tilde{q}_0$ (or $q_0$) must be greater than zero for the group velocity to be initially positive (northward).
Figure 1. Views from three perpendicular directions of the four possible rays emanating from one point. All four rays are seen in views (a) and (b), but in view (c) the rays defined by $k > 0$ appear directly on top of those defined by $k < 0$. $y_m = (f_m - f_0)/\beta + y_0$ is the coordinate of turning latitude.
the figure we have assumed, for convenience, that $\tilde{q}_0$ is the same magnitude for each curve and can be positive or negative. We can see all four rays in one view for the case (a) where we are looking straight down. $\tilde{q}_0 > 0$ delineates whether the ray will initially go (north) and $k \geq 0$ whether it will go (east). For case (b) where we look due north, the rays initially going north will appear to lie above those going south. For the case (c) looking due west, the rays going east and west appear to coincide.

We should keep in mind that this is only one set of a family of sets of rays emanating from every point on the surface. The sets are identical if we assume $f_0 = 1$, but originate from different points. The location of the turning latitude $y_m$ (or $f_m$) differs for rays from different latitudes, but the distance $y_m - y_0$ remains constant.

For the special case for $k = 0$, the four rays degenerate into two rays, one going north, the other south. These rays would look like a line along the $y$ axis terminating at $y_m$ for the case (a) in Figure (1), and like a line straight down for case (b). For case (c), they look like Figure (2) with a cusp formed at the turning point.

As stated previously, the radical in the denominator of (4.34) can become zero at a point on a path where $\tilde{q}_0 < 0$. We call this point the critical point. Equating the radical to zero, we find the critical point to be

$$t_c = \tilde{t} - \frac{1}{\tilde{\beta}} \left( \frac{k^2}{|\tilde{q}_0|} \right)^{1/3} (|\tilde{q}_0|^{2/3} - |k|^{2/3})$$  \hspace{1cm} (4.38)

in terms of the time, and

$$\xi_c = \frac{-\tilde{\beta}^2 (k^2 + \tilde{q}_0^2)^2}{3\tilde{\beta} f_0 \tilde{p}^3} - \tilde{q}_0$$  \hspace{1cm} (4.39)

in terms of the corresponding transformed depth, where $\tilde{q}_0$ is given by (4.37b). For $k = 0$, the critical point corresponds to the point of the cusp shown in Figure 2, the turning point where the $y$-component of the group velocity vanishes. For the general case ($k \neq 0$), the turning point does not correspond to the critical point unless $|k|$ happens to equal $|\tilde{q}_0|$.

There is a question of which sign to choose for the radical in (4.34) for $t > t_c$ (or $\xi > \xi_c$) for the $\tilde{q}_0 < 0$ case. The asymptotic solution breaks down in the neighborhood of the critical point, so a special expansion must be made. This was carried out for the case where $k = 0$ (Kroll, 1973), and it turns out that the principal value of the radical should be chosen. The boundary layer about $\xi = \xi_c$ was found to be of order $(2/\tilde{B}) (2\tilde{\gamma} f_0)^{1/2}$ in terms of the transformed depth. For realistic parameters, the extent of the boundary layer would be of order 1/10 of the bottom depth.

d. The Eulerian viewpoint. It is usually convenient to consider the solution in the Eulerian sense and express the amplitude entirely in terms of the field variables ($x, y, \xi, t$). But we cannot easily eliminate $x_0, y_0$, and $t_0$, algebraically from our equa-
tions. For the case where \( k = 0 \), this can be done rather easily, however.

Letting \( k = 0 \) in the equations (4.33), (4.34) and (4.35), we find for the two cases of \( \bar{q}_0 > 0 \) and \( \bar{q}_0 < 0 \) that the rays are given by

\[
y - y_0 = -\frac{\lambda}{\beta} \left[ \left( \frac{\xi - \xi_c}{\xi_c} \right)^{1/3} - 1 \right] \quad \text{for } \bar{q}_0 > 0
\]  
\[
y - y_0 = -\frac{\lambda}{\beta} \left\{ \begin{array}{l}
\left[ \left( \frac{\xi - \xi_c}{\xi_c} \right)^{2/3} - 1 \right] \xi < \xi_c \\
\left[ \left( \frac{\xi - \xi_c}{\xi_c} \right)^{1/3} - 1 \right] \xi > \xi_c
\end{array} \right. \quad \text{for } \bar{q}_0 < 0
\]  

the travel time of a disturbance to a depth \( \xi \) is

\[
t - t_0 = \left( t_0 - \frac{i}{k} \right) \left[ \left( \frac{\xi - \xi_c}{\xi_c} \right)^{1/3} - 1 \right] \quad \text{for } \bar{q}_0 > 0,
\]  
\[
t - t_0 = (t - t_0) \left\{ \begin{array}{l}
1 - \left( \frac{\xi_c - \xi}{\xi_c} \right)^{1/3} \xi < \xi_c \\
1 + \left( \frac{\xi - \xi_c}{\xi_c} \right)^{1/3} \xi > \xi_c
\end{array} \right. \quad \text{for } \bar{q}_0 < 0;
\]

the amplitude is

\[
V_0(y,\xi,t) = \frac{K(2\beta \lambda)^{3/4} \sqrt{F_0 F(\xi)} \bar{q} U(t_0(y,\xi,t))}{[3f_\xi \beta B(\xi - \xi_c)]^{5/6}} \quad \text{for } \bar{q}_0 > 0,
\]

and

\[
V_0(y,\xi,t) = \begin{cases} 
\frac{K(2\beta \lambda)^{3/4} \sqrt{F_0 F(\xi)} \bar{q} U(t_0(y,\xi,t)) U(t_0(y,\xi,t) - i)}{[3f_\xi \beta B(\xi - \xi_c)]^{5/6}} & \xi < \xi_c \\
-iK(2\beta \lambda)^{3/4} \sqrt{F_0 F(\xi)} \bar{q} U(t_0(y,\xi,t)) U(t_0(y,\xi,t) - i) & \xi > \xi_c
\end{cases} \quad \text{for } \bar{q}_0 < 0;
\]

and the frequency is

\[
\omega(y,\xi) = 1 + \beta y + \lambda \left( \frac{\xi - \xi_c}{\xi_c} \right)^{2/3} \quad \text{for } \bar{q}_0 > 0
\]

and
From (4.39) we have for $k = 0$,

$$
\xi_c = \frac{- (2 f_0 \lambda)^{3/2} \bar{q}_0}{3 f_0 \hbar \beta |\bar{q}_0|}.
$$

(4.44)

An alternate asymptotic method was developed from the Eulerian viewpoint for the special case where $l = k = 0$ (Kroll, 1973). In that development, integral transforms and the method of steepest descents were used. The result corresponds to the equation (4.42a) with $l = 0$. Even for the simplified case of $l = k = 0$, that method was much more difficult than the ray theory method.

We have already plotted the rays of equations (4.40a) and (4.40b) in Figure 2. For this special case, the ray trajectories are independent of time which is not true in general. Figure 3 shows the evolution of the amplitude field in time over the depth at a particular value of $y$ for the case $\bar{q}_0 < 0$. The character of the field is such that the source can send disturbances from the surface northward only within the time interval, 0 to $t$. Every disturbance produced within this interval reaches $\xi = \xi_c$ at exactly the same time, $t$, which can be seen from equation (4.41b). The result is the odd spatial pulse moving down with time as shown. We should keep in mind that our equations break down in the neighborhood of $\xi = \xi_c$ and that the amplitude does not actually become unbounded.

The evolution of the amplitude field for the general case, $l \neq 0$, $k \neq 0$, can be
surmised for $\tilde{q}_0 < 0$ also. We can see from equations (4.38) and (4.39) that $t_c$, the
time at which a disturbance will reach the critical point, and $\xi_c$, the transformed
depth of the critical point, both increase with $t_0$, the time at which the disturbance
is produced at the surface. So in this case, the "spike" moves down with time as
opposed to the case $k = 0$ where this point is fixed. This is illustrated in Figure 4.

We should keep in mind that Figures 3 and 4 represent only the case where $\tilde{q}_0 < 0$,
a northward family of rays. Eventually $\tilde{q}_0$ vanishes as the time, $t_0$, increases. In the
figures, $\tilde{q}_0 = 0$ represents the abrupt cut-off of the trailing edge of the downward
traveling pulse. As $\tilde{q}_0$ becomes positive, the rays are all southward. This field is not
shown.

**e. Summary of the ray solution results.** The propagation properties of a surface-
-generated disturbance from the point $x_0, y_0$ at the time $t_0$ are as follows:

1. The propagation is always symmetric in the $x$-direction. The magnitude of the
   $x$-component of the group velocity $\left( \frac{dx}{dt} \right)$ is constant with the direction being east-
   ward for $k > 0$ and westward for $k < 0$. It vanishes for $k = 0$.

2. The $y$-component of the group velocity $\left( \frac{dy}{dt} \right)$ is always southward for $\tilde{q}_0 > 0$ and
   initially northward for $\tilde{q}_0 < 0$. For the latter case, $\left( \frac{dy}{dt} \right)$ will eventually vanish at the
   turning point $t = \tilde{t}$, which corresponds to a turning latitude $f_m$, and become south-
   ward directed for $t > \tilde{t}$.

3. The magnitude of the group velocity increases with $\lambda$ and decreases with the
   horizontal wave numbers.

4. The ratio of the vertical to the horizontal propagation velocity, with these veloc:
   ities expressed in dimensional terms, is $0 \left( \frac{\sqrt{2f_0\lambda}}{F(\xi)} \right)$, generally a small quantity.

The properties of the amplitude are as follows:

1. In terms of the ray variables, the amplitude of a disturbance on a ray decreases
   with $\lambda$. But from an Eulerian point of view, looking at the field at a given location
   and time, the amplitude increases with $\lambda$. The difference here is due to the fact that
   the group velocity increases with $\lambda$. The amplitude from both points of view increases
   with the horizontal wave numbers.

2. The amplitude of southward rays decreases with the transformed depth, $\xi$. But, for
   northward rays, the amplitude becomes unbounded at the critical point $t = t_c$
   (or $\xi = \xi_c$). Except for the special cases where $k = 0$ or $|q_0| = |k|$, the critical point
does not correspond to the turning point.
The properties of the frequency are as follows:

1. In terms of the ray variables, the frequency is constant along a given ray. From an Eulerian standpoint, at a given latitude the frequency of the field increases with depth for a family of southward rays. For a family of northward rays, the frequency decreases until \( \xi \) reaches a point corresponding to a turning point of one of the family of rays. The the frequency increases as \( \xi \) increases.

2. The minimum frequency that can occur at any depth at a given latitude corresponds to the frequency of the ray which happens to have a turning point at that latitude. If the local inertial frequency at a given latitude is \( f \), then the minimum frequency of the inertial motion in the interior is \( \omega = f + \lambda \left( 1 - \frac{g^2}{\eta \omega} \right) \), using (4.36) in (4.37e). For \( k = 0 \), the minimum is \( f \) itself.

It is important to keep in mind that the character of the forcing from the boundary layer is crucial to the character of the solution. It is through the forcing that the parameter \( \lambda \) and the horizontal wave numbers are introduced into the interior system. The ray trajectories, the propagation velocity, and the amplitude depend strongly on these quantities, with no propagation at all as \( \lambda \) vanishes. So clearly these quantities, especially \( \lambda \), are important.

Another important aspect of the forcing is the fact that the forcing frequency is changing with latitude and is a small amount \( \lambda \) greater than the local inertial frequency at every latitudinal point. This results in both the turning latitude and critical point being located at different latitudes for rays originating from different latitudes.

As a contrasting example, consider what happens if we had used a frequency, \( \omega_c \), independent of latitude for the case where \( k = 0 \). Then the turning latitude (and the latitude of the critical point) would be the same for all rays originating at latitudes whose local inertial frequency was less than \( \omega_c \). For this kind of forcing, we would expect a large amplitude field at this latitude, independent of the location and spatial distribution of the forcing. This result is not possible for our forcing.

5. The determination of the parameter \( \lambda \)

In Section 1, we made the assumption that the forcing from the boundary layer was a small amount \( \lambda \) greater than the local inertial frequency. We will now justify this assumption and evaluate \( \lambda \) by including the pressure gradients in the boundary layer analysis.

It is not the purpose of this paper to analyze the complicated motion in the surface layer. In reality, if we go beyond the lowest order we might expect non-linear effects to be at least as important as the pressure perturbation. What we will illustrate in the following is that, at least for the linearized, stratified boundary layer, there exists a mechanism which tends to produce a frequency slightly greater than the local inertial frequency.
We consider the following linearized equations in the boundary layer (dropping overbars),

\[
\begin{align*}
\frac{u_t - fv}{u_{zz}} & = E_{uz}, \\
\frac{v_t + fu}{-P_y + E_{vzz}} & = 0, \\
\frac{P_z}{-\varrho} & = 0, \\
\frac{\varrho_t - B^2 w}{E_\varrho_{zz}} & = 0, \\
\frac{v_y + w_z}{0} & = 0.
\end{align*}
\]

(5.1)

Here we have assumed for simplicity that the forcing is uniform in \(x\) \((k = 0)\) so that we can neglect derivatives in \(x\). We assume that \(B\) is constant in the boundary layer, and we modify one of our original assumptions and include the vertical diffusion of density with a diffusivity equal to that of momentum.

We now apply the following Fourier transforms in \(z\), \((\tilde{u}, \tilde{v}, \tilde{P}) = \int_0^\infty (u, v, P) \cos pz dz\) and \((\tilde{\varrho}, \tilde{w}) = \int_0^\infty (\varrho, w) in pzdz\). We assume that all quantities vanish as \(z \to \infty\) and use the boundary conditions at \(z = 0\): \(w = 0, v_z = 0, u_z = \frac{\tau_0 D}{\bar{V}_0 \varrho} e^{-\beta y} \delta(t), \) and \(\varrho = 0\).

Using the transformation \((\tilde{u}, \tilde{v}, \tilde{P}, \tilde{q}, \tilde{w}) = (U, V, P, R, W) e^{-\beta y},\) we find after eliminating everything but \(V,\)

\[V_{tt} + f^2 V - \alpha^2 V_{yy} = 2f Ke^{-\beta y} \delta(t),\]

(5.2)

where \(K = \frac{\tau_0}{2\bar{V}_0 \varrho_0 Df_0}\) and \(\alpha = \frac{B}{p}\).

Without loss of generality, we specify \(l \geq 0\). Then \(V\) divides into two parts, \(V_1 + V_2\), which represent wave solutions traveling in opposite directions with slightly differing characteristics due to the \(\beta\) effect. Then we have

\[V_{nt} + f^2 V_n - \alpha^2 V_{nyy} = f Ke^{-\beta y} \delta(t) + (-1)^n \gamma(t) \quad (n = 1, 2)\]

(5.3)

where \(\gamma(t)\) is an arbitrary function to be chosen at our convenience since it vanishes for \(V = V_1 + V_2\). To solve (5.3) asymptotically, we assume new “times”, \(t_1 = g_1(t)\) and \(t_2 = g_2(t)\), which will be determined assuming \(g_n(0) = 0\). Then we have

\[V_{n_{tt}} + \frac{g_n''(t_n)}{[g_n'(t_n)]^2} V_{n_{tt}} + \frac{1}{[g_n'(t_n)]^2} \left[f^2 V_n - \alpha^2 V_{n_{yy}}\right] = f Ke^{-\beta y} \delta(t_n) + (-1)^n \Gamma_n(t_n),\]

(5.4)

where \(\Gamma_n(t_n) = \frac{\gamma(t)}{[g_n'(t_n)]^2}\). We have used the property of the delta function that \(\delta(t) = g_n'(0) \delta(t_n)\).

We now assume a solution in the boundary layer of the form

\[V_n = q_n(y, t) e^{-\beta y} (-1)^n \delta(t) \quad (n = 1, 2).\]

(5.5)
We then assume that
\[(l + (-1)^n \beta t_n)^2 \gg \left| \frac{\varphi_{nyy}}{\varphi_n} - 2i(l + (-1)^n \beta t_n) \frac{\varphi_{ny}}{\varphi_n} \right|,\]
so that we can use the approximation
\[V_{nyy} \approx -(\bar{l} + (-1)^n \beta t_n)^2 V_n. \tag{5.6}\]
Thus we obtain an ordinary differential equation in \(t_n,\)
\[V_{nt_n t_n} + \frac{g_n''}{(g_n')^2} V_{nt_n} + \left[ \frac{f^2 + \alpha^2(l + (-1)^n \beta t_n)^2}{(g_n')^2} \right] V_n = \frac{f Ke^{-ity}}{(g_n'(0))} \delta(t_n) + \]
\[+ (-1)^n \Gamma_n(t_n) \quad (n = 1, 2). \tag{5.7}\]
We call \(t_n\) the “fast” time, and we define a “slow” time as \(\tau_n = \alpha \beta t_n\) where \(\alpha\) is assumed small (i.e., \(p\) is large). For the moment, we will assume \(\alpha \beta t_n\) and \(t\) are 0(1).

We now use the adiabatic invariant perturbation method as outlined by Cole (1968) to solve equation (5.7). For the fast time scale, we want the frequency to be \(f\) (constant in time). So we set the term in the brackets in (5.7) equal to \(f^2\) which determines \(t_n = g_n(t)\). Using the condition \(g_n(0) = 0\), we derive
\[t_n = g_n(t) = \frac{f}{\beta \alpha} \sinh \left[ \frac{\alpha \beta t}{f} + (-1)^n \sinh^{-1} \left( \frac{\alpha \beta t}{f} \right) \right] - (-1)^n \bar{l} / \beta. \tag{5.8}\]
From (5.7) we then have
\[V_{nt_n t_n} + \alpha G_n(\tau_n) V_{nt_n} + f^2 V_n = \frac{f Ke^{-ity}}{g_n'(0)} \delta(t_n) + (-1)^n \Gamma_n(t_n), \tag{5.9}\]
where
\[G_n(\tau_n) = \frac{(\tau_n - (-1)^n \alpha l)}{f^2 + (\tau_n - (-1)^n \alpha l)^2}.\]
We now expand \(V_n\) in powers of \(\alpha\) such that
\[V_n = V_n^0(t_n, \tau_n) + \alpha V_n^1(t_n, \tau_n) + \ldots \tag{5.10}\]
Since \(\tau_n = \beta \alpha t_n\), (5.9) then becomes
\[V_n^0_{nt_n t_n} + \alpha(2 V_n^0_{nt_n t_n} + V_n^1_{nt_n t_n} + G_n(\tau_n) V_n^0_{nt_n}) + \ldots \]
\[+ f^2(V_n^0 + \alpha V_n^1 + \ldots) = \frac{f Ke^{-ity}}{g_n'(0)} \delta(t_n) + (-1)^n \Gamma_n(t_n). \quad \tag{5.11}\]
The quantity \(g_n'(0)\) is known and can be calculated from (5.8):
which is independent of \( n \), implying we can drop the \( n \). Though \( \gamma(t) \) must be independent of \( n \), \( \Gamma_n \) is not, and must be ordered in \( \alpha \) also.

For the 0th order we have

\[
V_{n_0}^0 + f^2 V_{n_0}^0 = \frac{f Ke^{-il}\gamma}{\tilde{g}} \delta(t_n) + (-1)^n \Gamma_n^0(t_n).
\]  

(5.13)

If \( \Gamma_n^0(t_n) = \frac{if Ke^{-il}\gamma}{\tilde{g}} \delta'(t_n) \), the solution to (5.13) is straightforward. To obtain this value of \( \Gamma_n^0 \), we let \( \gamma(t) = \frac{if Ke^{-il}\gamma}{\tilde{g}} \delta'(t) \), and using the relation \( \varphi(x) \delta'(x) = \varphi(0) \delta(x) - \varphi'(0) \delta(x) \), obtain

\[
\Gamma_n(t_n) = \frac{if Ke^{-il}\gamma}{\tilde{g}} [\delta'(t_n) + (0(\alpha^2) \delta(t_n)].
\]  

(5.14)

This gives us the desired expression for \( \Gamma_n^0 \). We then can solve (5.13) to obtain

\[
V_n^0 = \lim_{\varepsilon \rightarrow 0} (-1)^n \frac{i Ke^{-il}\gamma}{\tilde{g}} e^{(-1)^n i f t_n} U(t_n - \varepsilon) + A_n(\tau_n) e^{i f t_n} + B_n(\tau_n) e^{-i f t_n},
\]  

(5.15)

where we have assumed \( \delta(t_n) = \lim_{\varepsilon \rightarrow 0} \delta(t_n - \varepsilon) \) to overcome difficulties at \( t_n = 0 \). We have four arbitrary constants and only one second-order equation with two initial conditions: \( V(0) = V_\tau(0) = 0 \). We must end up with the form assumed in (5.5) and that implies that we should set \( B_1(\tau_1) = A_2(\tau_2) = 0 \).

We add \( V_1^0 \) and \( V_2^0 \) to obtain \( V_0 \). Using the initial conditions, we find \( A_1(0) = B_2(0) = 0 \). To find \( A_1(\tau_1) \) and \( B_2(\tau_2) \), we go to the next order of (5.11). At order \( \alpha \) we have

\[
V_{n_1}^1 + f^2 V_{n_1}^1 = - G_n(\tau_n) V_{n_1}^0 - 2 V_{n_1}^0.
\]  

(5.16)

Letting \( n = 1 \), we have

\[
V_{n_1}^1 + f^2 V_{n_1}^1 = - G_1(\tau_1) i f e^{il\tau_1} \left\{ Ke^{-il\gamma} i \frac{i g}{i g} + A_1(\tau_1) \right\} - 2 i f \frac{d A_1(\tau_1)}{d \tau_1} e^{il\tau_1}.
\]  

(5.17)

To eliminate secular terms, we set the right hand side of (5.17) to zero. Using \( A_1(0) = 0 \), we then have

\[
A_1(\tau_1) = -\frac{K}{i g} \left\{ 1 - \left[ \frac{(f^2 + \alpha^2 l^2)^{1/4}}{(f^2 + (\tau_1 - \alpha l)^2)^{1/4}} \right] e^{-il\gamma}. \right\}
\]  

(5.18)

Then from (5.15), evaluating \( \tilde{g} \),

\[
V_{1}^0 = \frac{- K e^{il\tau_1 - il\gamma}}{i [f^2 + \alpha^2 l^2]^{1/4} [f^2 + (\tau_1 - \alpha l)^2]^{1/4}}.
\]  

(5.19)
and, inverting the transform, we have

\[ v_1 \approx \frac{2K}{i\pi} e^{-i\gamma} \int_0^\infty e^{-Ept} \cos pz \cdot \exp \left\{ f^2_p \sinh \left( \frac{B\beta t}{pf} - \sinh^{-1} \left( \frac{Bl}{fp} \right) + \frac{fl}{\beta} \right) \right\} \frac{dp}{1 + \left( \frac{B^2 f^2}{f^2 p^2} \right)^{1/4}} \left[ 1 + \sinh^2 \left( \frac{B\beta t}{fp} - \sinh^{-1} \left( \frac{Bl}{fp} \right) \right) \right]^{1/4}, \tag{5.20} \]

where we have eliminated \( t_1 \) and \( \tau_1 \) in favor of \( t \).

Integrating (5.2a) is prohibitive. We have been assuming \( \alpha \beta t \) and \( \alpha l \) are \( O(1) \). If we now assume that these quantities are also small, then we can simplify (5.20) considerably. This should be a valid assumption if the assumption that \( \alpha \) is small is valid, since \( \beta t \) = \( O(1) \) and \( l \) = \( O(1) \) are reasonable. We will consider the justification for \( \alpha \) small later.

So we let \( \frac{B\beta t}{pf} \) and \( \frac{Bl}{fp} \) be much less than 1. Then we have

\[ \sinh^{-1} \left( \frac{Bl}{pf} \right) \approx \frac{Bl}{pf} - \frac{1}{6} \left( \frac{Bl}{pf} \right)^3, \]

and

\[ \sinh \left( \frac{B\beta t}{pf} - \sinh^{-1} \left( \frac{Bl}{pf} \right) \right) \approx \frac{B}{pf} \left( \left( \beta t - l \right) + \frac{1}{6} \left( \frac{B}{pf} \right)^2 \left( \left( \beta t \right)^2 - 3l(\beta t)^2 + 3\beta tl^2 \right) \right). \]

We then express the integrand of (5.20) to the lowest order in \( \alpha \) except in the eiconal part where we assume the next higher order of \( \alpha \) is important in finding the small increase in frequency we are looking for. We then have

\[ v_1 \approx \frac{2K}{i\pi} e^{-i\gamma} \int_0^\infty e^{-Ept} \exp \left[ it \left( f + \frac{1}{6} \frac{B^2}{fp^2} N_1(t,l) \right) \right] \cos pz dp \tag{5.21} \]

where \( N_1(t,l) = (\beta t)^2 - 3\beta tl + 3l^2 \). For \( z = 0 \) we can solve (5.21) using the transformation \( q = 1/p^2 \). This yields the tabulated form (Oberhettinger, 1957)

\[ v_1(0) \approx \frac{K}{i\sqrt{\pi E t}} \int_0^\infty q^{-3/2} e^{-\frac{Et}{q}} \exp \left[ \frac{ft}{6} + \frac{1}{f} \frac{B^2}{N_1 q} \right] dq, \]

which yields

\[ v_1(0) \approx \frac{K}{i\sqrt{\pi E t}} e^{-\lambda_1 t} e^{i(f + \lambda_1 t - \psi)}, \tag{5.22} \]

where \( \lambda_1(t,l) = \left( \frac{1}{3} \frac{B^2}{f} N_1(t,l) \right)^{1/2} \). Using \( B = \delta F_0 \) and \( \beta = \frac{L}{R} \cot \varphi_0 \), we can express \( \lambda_1 \) in the more calculable form,
\[
\lambda_1(t,\bar{t}) = F_0 E_\mathcal{R}^{1/2} \left( \cot \varphi_0 \right) \left( \frac{\bar{N}_1(t,\bar{t})}{3f} \right)^{1/2},
\]
\[(5.23)\]

where \( E_\mathcal{R} = \frac{\nu}{R^2 f_0} \), \( \bar{N}_1(t,\bar{t}) = t^2 - 3t\bar{t} + 3\bar{t}^2 \) and \( \bar{t} = \frac{t}{\beta} \).

To determine the validity of our assumption of \( \alpha = B/p \) small, we could try to integrate the next order term in the expansion in \( \alpha \). But this would be difficult. An easier, though less conclusive method, is to examine the saddle point assuming the method of steepest descents is applicable. For \( z = 0 \), we find the magnitude of the saddle point of (5.21) to be given by \( |p_2| = \left( \frac{B^2}{6f} \right)^{1/4} \left( \frac{N_1}{E} \right)^{1/4} \left( \frac{F_0}{6f} \right)^{1/4} \left( \frac{\bar{N}_1}{E_\mathcal{R}} \right)^{1/4} \). For \( F_0 = 0(10^2), E_\mathcal{R} = 0(10^{-12}) \) and \( t \sim \bar{t} = 0(10^2) \), this implies \( \alpha = \frac{B}{p} = 0(10^{-3}) \) which is indeed small.

Going back to (5.16) and considering \( n = 2 \), it can be shown that
\[
\nu_2(0) \approx \frac{-K}{i\sqrt{\pi E_t}} e^{-\lambda_2 t} e^{-i((\beta t) + \lambda_2) t + b t},
\]
\[(5.24)\]

where \( \lambda_2(t,\bar{t}) = \left( \frac{1}{3} \frac{E}{f} N_2(t,\bar{t}) \right)^{1/2} \) and \( N_2(t,\bar{t}) = (\beta t)^2 + 3\beta t\bar{t} + 3\bar{t}^2 \). Again a more calculable form is
\[
\lambda_2(t,\bar{t}) = F_0 E_\mathcal{R}^{1/2} \left( \cot \varphi_0 \right) \left( \frac{\bar{N}_2(t,\bar{t})}{3f} \right)^{1/2},
\]
\[(5.25)\]

where \( \bar{N}_2(t,\bar{t}) = t^2 + 3t\bar{t} + 3\bar{t}^2 \). Adding \( \nu_1(0) \) and \( \nu_2(0) \), we have
\[
\bar{\nu}(0) \approx \frac{K e^{-i\lambda y}}{i\sqrt{\pi E_t}} \left[ e^{-\lambda_2 t} e^{i((\beta t) + \lambda_2) t} - e^{-\lambda_2 t} e^{-i((\beta t) + \lambda_2) t} \right],
\]
\[(5.26)\]

where we have reintroduced the overbar for a boundary layer quantity.

If \( \lambda_1 = \lambda_2 = 0 \), then
\[
\bar{\nu}(0) \approx \frac{2K}{\sqrt{\pi E_t}} e^{-i\lambda y} \sin ft.
\]
\[(5.27)\]

We take the imaginary part of (3.3) and let \( Y(x,y) = \frac{\tau_0 e^{-i\lambda y}}{\bar{V}_0 Df_0} = 2Ke^{-i\lambda y} \), as was done to obtain (3.6) where we had \( x \) dependence. We then see that (5.26) is the zero-order solution for \( \nu \) at \( z = 0 (\xi = 0) \). We can show from this result and (3.4) that
\[
\bar{w}(z = 0) = \bar{\nu}_y(0) \sqrt{\pi E_t}.
\]
\[(5.28)\]

We assume that (5.28) is approximately valid for the more complicated solution for \( \lambda_1, \lambda_2 \neq 0 \). Thus the Ekman suction velocity to an order \( E \) perturbation becomes
\[
\omega^0 = -\bar{w}(z = 0) \approx -K \left[ (\beta t - l) e^{-\lambda_1 t} e^{i((\beta t) + \lambda_1) t - b \bar{t}} + (\beta t + l) e^{-\lambda_2 t} e^{-i((\beta t) + \lambda_2) t + b \bar{t}} \right],
\]
\[(5.29)\]
The correspondence between this result and (3.6) is clear. The driving frequency has been slightly increased by the factors \( \lambda_{1,2} \geq 0 \), as we have been assuming, and there is eventual exponential decay. For the realistic values, \( F_0 \sim 10^2, E_R \sim 10^{-12}, L = L_y \sim 100 \text{ km} \) and, assuming \( \beta t \ll l = 2\pi \), we have \( \lambda_1 \approx \lambda_2 \approx .04 \). The slow decay and the variation of \( \lambda_{1,2} \) with time and space may be included in our ray solution in the manner explained after equation (4.32).

As we said at the beginning of this section, this calculation may have quite limited applicability because of the exclusion of non-linear effects. But I think it does give justification to the assumption that the Ekman suction of the boundary layer forces the interior at a frequency slightly greater than the local inertial frequency.

6. The effects of finite depth, realistic stratification, intermittent forcing, and phase mixing

The real ocean, of course, has a finite depth. The theory so far, though we have introduced the bottom depth, is independent of this parameter. The reflection from a level bottom is quite easily dealt with using symmetry. We consider the forcing of the system to be symmetric about the bottom, as shown in Figure 5 for just one of the family of rays coming from the surface.

To calculate the reflected ray, we merely calculate for the ray going past the bottom \((z = 1)\) into a region where the variation of the stratification \( F(z) \) is a mirror image about \( z = 1 \). We then visualize this ray as coming up from the bottom. We could visualize a succession of symmetric sources to deal with succeeding reflections, but we will not consider more than the one reflection from the bottom.

The reflection properties of a ray from an arbitrarily angled surface can be calculated. The reflection of rays from a vertical southern boundary has been calculated (Kroll, 1973). But no more reflections will be considered here.

Figure 6 shows a comparison of ray trajectories for realistic \((F_2)\) and linear \((F_1)\) stratification. The horizontal travel for the path associated with \( F_2 \) is greater near the surface than that associated with \( F_1 \). This is due to the thermocline of \( F_2 \). But the difference is not gross because effects of stratification are integrated through the transformed depth, \( \xi = \frac{1}{F_0} \int F(z) \, dz \).

However, if we look at the vertical wave number,

\[ m = \frac{\delta k}{\delta z} = \frac{\delta}{} \]
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assumed the forcing to be impulsive in time and sinusoidal in space. For general forcing in space and time, the total solution, $v_T$, is given by

$$v_T(x,y,\xi,t) = \frac{1}{(2\pi)^2} \int \int \int \tilde{G}(k,l,t-t') V(k,l,t',x,y,\xi) e^{iS(k,l,t-t',x,y,\xi)} dk dl dt'$$

(6.2)

where $\tilde{G}$ is the Fourier transform in $x$ and $y$ of the forcing.

To say anything about the observed intermittency of inertial motion in the deep, we must say something about this integral. This is because both the effect of real forcing in time and the summation over wave number space can contribute to intermittency.

Pollard and Millard (1970) have shown that the intermittency of inertial motions in the mixed layer depends strongly on the intermittency of the wind, not only on the waxing and waning of the wind amplitude, but more importantly, on the intermittent changing of direction. The oscillations can be enhanced or reduced very quickly depending on the changes in relation to the phase of motion. The implication for deep motions is apparent. If the oscillations produced in the boundary layer are intermittent, then the resulting downward propagation will be intermittent.

The integration over wave number space, phase mixing, can cause decay which will also result in intermittency. An idea of what to expect was found by integrating (6.2) in an approximate manner assuming $k = 0$, $l \gg \beta t$ and $\xi \ll |\xi|_c$ (Kroll, 1973). It was found that the decay went approximately as $e^{-\gamma y}$ where $\gamma = F_0 E_R^{1/2} \cot \varphi_0$, and $\gamma$ is a measure the scale of the Gaussian-type spatial distribution of the forcing ($G(y) = \sqrt{2\pi} e^{-y^2/2} e^{-\gamma (\beta y)^2}$). For forcing having a spatial scale of 0(100 km), $\gamma$ is 0(10^3). For $F_0 = 0(10^2)$ and $E_R = 0(10^{-12})$ the $e$-folding time of decay was found to be on the order of a couple of weeks at mid-latitudes. This would produce intermittency having a considerably longer duration than that normally observed. This tends to indicate that the intermittency of the wind is more important.

Another possible source of intermittency is the effect of the summation of the incident and bottom reflected fields. Each has slightly differing frequencies at a given
depth which when added together causes sinusoidal beats that can be interpreted as intermittency. The frequency of the beats will increase as the difference in the frequencies increases. So we would expect the least intermittency due to this phenomenon near the bottom, where the frequencies of the fields are closest to one another. The variation of the frequencies with depth of the fields associated with southward and northward families of rays is shown in Figure 13 in Section 7.

7. The application of the theory to real ocean conditions

Over a number of years, the Woods Hole Oceanographic Institution has monitored a fixed vertical array of current meters at Site D, 39°N, 70°W in the North Atlantic. In what follows, we use the values of parameters appropriate to this site in our theory, and attempt to compare these results with actual data.

Figure 7 shows the seasonal variation of the depth profiles of the stratification (Brunt-Väisälä frequency) at Site D. The stratification near the surface varies considerably, with the summer-autumn values being 30 times or so greater than the winter-spring values. The relation we use to evaluate $A$ in succeeding calculations is given by

$$\lambda = F_0 \sqrt{\frac{v}{L^2f_0}} \sqrt{1 + k^2}. \quad (7.1)$$

This is derived by assuming $l \gg t$ in either of equations (5.23) or (5.25) and estimating that $k$ comes into the expression as shown. The value of $\lambda$ then varies linearly with $F_0$, the stratification at the surface. So we deduce from equation (4.42) that the amplitudes in summer-autumn should be significantly greater than those in winter-spring assuming comparable forcing. But this kind of difference is not discernible in any data that I have seen.

At this point we should keep in mind that in our analysis of the boundary layer, we assumed $F$ to be constant throughout the layer. Our results are no longer valid if $F$ varies greatly in this layer as it does at Site D in winter and spring. The fact that there seems to be no observable difference between seasons may indicate the following possibilities: (1) In the case in which $F_0$ is small, the energy may diffuse to a point at which $F$ is large enough to produce a $\lambda$ that allows a significant amount of propagation. This would make the value of $F_0$ effectively equivalent to that of the bottom of the mixed layer. (2) As we have speculated in Section 5, there may be strong non-linear effects on $\lambda$ since the mixed layer is a strongly turbulent region. My guess is that the second possibility is most likely, but we will not resolve the issue here.

We use the autumn-summer stratification since this condition best fulfills our assumption of $F$ essentially constant ($F_0$) near the surface. We then have

$$F(z) = \begin{cases} 340 & 0 \leq z \leq 0.007 \\ 11z^{-0.62} & 0.007 \leq z \leq 1 \end{cases}. \quad (7.2)$$
where $F_0 = 340$ and $\hat{F} = \int F(z)\,dz = 29$. For the other pertinent parameters, we use

$$\tau_0 / \theta = 2 \left( \frac{\text{cm}^2}{\text{sec}} \right), \quad D = 2600 \text{ m}, \quad v = 10 \left( \frac{\text{cm}^2}{\text{sec}} \right), \quad \text{and} \quad f_0 = 0.92 \times 10^{-4} \text{sec}^{-1}.$$

We are most interested in field solutions because the data from the current meters must be interpreted in an Eulerian sense. As a consequence, most of the calculations are for the case where $k = 0$ ($L_x \to \infty$) because the field solution is readily available (equations (4.40)-(4.43)). However, computer calculations were made for the field solutions for the general case (equations (4.33)-(4.36)) with $L_x = -L_y = 100$ km and $L_x = L_y = 100$ km. These limited calculations hardly exhaust the possibilities, but they do give an idea of what to expect.

Figure 8 shows the amplitude field for various wavelengths for the southward rays. This would be a snapshot taken at $t = 100$ ($\sim 13$ days). We see that the character of the field for $L_x = 100$ km is the same as when $L_x = \infty$. For a realistic forcing structure of $L_x \sim L_y \sim 0$ (100 km), the amplitude is realistic at the bottom, 0 (1 cm/sec), with the larger amplitudes being associated with smaller wavelengths.

Figure 9 also shows the amplitude field, but considers a reflection from the bottom as well. On this and subsequent figures with reflection, we show the incident and reflected parts and not their sum. Here we show the effect of the removal of the stratification. We see that for the northward rays ($L_y > 0$), $|V| / |\hat{F}|$ increases with depth.
Figure 9. A comparison at a particular point \((x,y)\) between the amplitude fields generated by southward (S) and northward (N) rays. One reflection from the bottom, shown by the stippled region, is considered in the variation with dimensionless depth \((z)\). The short dashed line represents the depth of the critical point for \(k = 0 (L_x = \infty)\). \(|V|\) is given by solid lines, and \(|V|/F\) by long dashed lines. The curves represent the conditions: a) \(L_y = -25\) km, \(L_x = \infty\); b) \(L_y = -50\) km, \(L_x = \infty\); c) \(L_y = -100\) km, \(L_x = \infty\); d) \(L_y = -L_x = -100\) km; e) \(L_y = -200\) km, \(L_x = \infty\); f) \(L_y = -300\) km, \(L_x = \infty\); g) \(L_y = -400\) km, \(L_x = \infty\).

Figure 10. The travel time \(t_T\) of the initial pulse \((t_0 = 0)\) to a given dimensionless depth \((z)\) for the southward rays. The curves represent the conditions: a) \(L_y = -25\) km, \(L_x = \infty\); b) \(L_y = -50\) km, \(L_x = \infty\); c) \(L_y = -100\) km, \(L_x = \infty\); d) \(L_y = -L_x = -100\) km; e) \(L_y = -200\) km, \(L_x = \infty\); f) \(L_y = -300\) km, \(L_x = \infty\); g) \(L_y = -400\) km, \(L_x = \infty\).

For \(L_x = \infty\), the critical point, \(\xi_c\), is reached within our view. For \(L_x = 100\) km, \(\xi_c\), which deepens with time, is not within our view (more reflections are necessary). The solution for \(L_x = \infty\) at \(\xi_c\) is finite because we have used the results of a solution in a boundary layer about \(\xi_c\) (Kroll, 1973).

Figures 10, 11, and 12 show properties of the travel time of a disturbance. We see from Figure 10 that a disturbance going southward with a wavelength of order 100 km will take about a week to reach the bottom. Also the travel time is shorter for the southward rays than for the northward rays as shown on Figure 11. Figure 12 shows that the smaller wavelengths propagate the fastest. This is due mainly to the dependence of \(\lambda\) on wavelength.

Figure 13 shows how the observed frequency of inertial oscillations will change with depth where \(\Delta \omega\) is the difference between the actual frequency \((\omega)\) and the local inertial frequency \((f)\). For \(L_x = \infty\), \(\Delta \omega\) vanishes at one point for the northward rays. In general, where \(L_x\) is finite, we have \(\Delta \omega \geq f_m - \omega_0(y_0(y,\xi,t)) \geq 0\) where \(f_m\), the turning latitude, is found from (4.36) and \(\omega_0 = f_0(y_0(y,\xi,t)) + \lambda\). For these calcula-
Figure 11. A comparison of the travel time of the initial pulse \((t_0 = 0)\) to a dimensionless depth \((z)\) for southward (S) and northward (N) rays. The stippled region shows one reflection from the bottom. The horizontal dashed line represents the depth of the critical point for \(k = 0 (L_x = \infty)\). The curves represent the conditions: a) \(L_y = -100\) km, \(L_x = \infty\); b) \(L_y = -L_x = -100\) km; c) \(L_y = L_x = 100\) km; d) \(L_y = 100\) km, \(L_x = \infty\).

Figure 12. The travel time, \(t_T\) (days), to the bottom of the initial pulse \((t_0 = 0)\) for various wavelengths for the southward rays for \(k = 0 (L_x = \infty)\).

Figure 14 shows the variation of the vertical wavelength \((L_z)\) with depth. We have used equation (6.1) with \(m = \frac{2\pi}{L_z} D\), assuming \(t = 0\) in \(\vec{q}_0\). It is clear in this case that the \(z\) dependence will be solely from the stratification. For horizontal wavelengths of 100 km, \(L_z\) is as short as 90 m at the surface and as long as 2900 m at the bottom.

In the introduction we listed the major properties of observed inertial motion. All of these properties are consistent with our theory. Of these, (b), (c), and (d) are the most important. Figures 8 and 9 confirm (b) that we can expect amplitudes of 0(1 cm/sec) at a depth of two kilometers. Concerning (c), the observed lack of vertical coherence of the frequency as compared with the greater horizontal coherence is consistent with rays of constant frequency traveling an order of 100 km horizontally for every kilometer downward. Equation (4.34) shows that the velocity amplitude varies with \(\left( F = \frac{N}{f_0} \right)^{1/2}\), confirming (d). This was previously derived by Munk and Phillips (1968) which indicates the applicability of a WKBJ approximation to which ray theory is related.

The demodulated data shown in Figures 15 and 16 were not gathered to test our theory. Most importantly, we lack any information about the wind. However, there are gross features which can be used for comparison.
Figure 13. A comparison at a particular point $(x,y)$ of the variation of the fractional change of frequency, $\Delta \omega / \omega$, of the field with depth between southward (S) and northward (N) rays. The stippled region shows one reflection from the bottom. The horizontal dashed line represents the depth of the critical point for $k = 0$. The curves represent the conditions: a) $L_y = 100$ km, $L_x = \infty$; b) $L_y = -L_x = 100$ km; c) $L_y = -100$ km, $L_x = \infty$; d) $L_y = L_x = -100$ km.

Figure 14. The dimensionless vertical wavelength, $(L_z/D)$ vs. dimensionless depth $(z)$ for various horizontal wavelengths, $L_x$ and $L_y$. The curves represent the conditions: a) $L_y = -L_x = -25$ km; b) $L_y = -L_x = -50$ km; c) $L_y = -L_x = -100$ km; d) $L_y = -L_x = -200$ km.

The properties (a), (b), and (e) listed in the introduction can be seen in these figures. Also, in Figure 15 we see that there are instances when the frequency is less than the local inertial frequency. The fact that this occurs relatively near the surface is consistent with our theory. Frequencies equal to or less than inertial should not propagate downward. The generation of frequencies less than inertial cannot be explained by our boundary layer analysis, however.

Figure 15 does not show any discernible trend in the variation of $\Delta \omega / \omega$ with depth. This would be consistent with a mixture of southward and northward rays. Nor is there any evidence of an event traveling straight downward. This too is consistent with theory.

The most important feature to be seen from Figure 15 is that there is a great deal of similarity between the phase plots for the deepest depths at 1044 m and 2066 m. But there is little similarity between any other pair of depths. This indicates that there could be a filtering process which inhibits much of the activity near the surface from propagating downward. This is consistent with our conclusion that $\lambda$ must be sufficiently large for significant downward propagation to occur. Those disturbances which do penetrate deeply seem to have a similarity at the two deep meters consistent with a family of rays. The frequency at 2066 m is slightly less than the frequency at 1044 m. This indicates that we might visualize disturbances propagating on a northward family of rays from the surface down to these depths.
Figure 15. The phase of the north-south velocity component of inertial motion at five depths derived from Site D data, beginning on 10/7/69. The fractional change of frequency \( \frac{\Delta \omega}{f} \) is represented by the slope of these phases as shown in the key. This reduced data has been derived by a de-modulation process about a period of 19.0 hrs.

The type of comparisons we have made with data cannot conclusively prove the applicability of our theory. A definitive experiment would consist of a horizontal array of current meter strings distributed in the north-south direction over a lateral extent of the order of 100 km. Wind conditions at the surface would have to be accurately monitored along the extent of the array and an order of 100 km north and south of it. Hopefully in this way one would be able to see a clearly defined disturbance travel laterally and downward from the surface as predicted.

8. Conclusions

The theory is consistent with the major features of observed inertial motion. Most importantly, it provides an explanation of the relatively large amplitudes observed at great depth.

For realistic conditions of wind forcing, horizontal velocity amplitudes of the
order of 1 cm/sec are predicted in the deep ocean. It is also predicted that a disturbance from the surface will take about a week to reach the bottom.

The parameter $\lambda$, the difference between the forcing frequency of the boundary layer and the local inertial frequency, is found to be quite important. Significant downward propagation from the boundary layer is not possible unless $\lambda$ is sufficiently large. The linear model used to evaluate $\lambda$ is probably not sufficiently accurate, and an improved model for $\lambda$ may be necessary.

Future work must include the investigation of the total solution, the integral over wave number space. Also an investigation of effects of a steady state current on the propagation would be profitable.

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