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ABSTRACT

Taylor's tidal problem for a rotating semi-infinite canal is reformulated to allow for the possibility of imperfect reflection of Kelvin waves. It is found that, just below the critical period at which Poincare mode propagation becomes possible, this mode becomes the principal energy-reflection mechanism. This critical period can easily exceed the tidal period, resulting in highly asymmetric oscillations.

The solutions are obtained by using collocation at the closed end of the canal; this method is found to be successful. The behavior for the limiting cases of a narrow canal and of a short-period oscillation is also discussed.

1. Introduction. This study is a reformulation of a problem first considered by Taylor (1921) and reconsidered by Defant (1925: see Defant 1961: 202–219). It concerns the reflection of Kelvin waves from the closed end of a semi-infinite rectangular canal. Cylindrical Poincaré waves radiated from the two corners (Packham and Williams 1968) may or may not carry energy away from the closed end.

Taylor and Defant considered two Kelvin waves of equal amplitude traveling in opposite directions in the canal; the origin of coordinates was chosen such
that their phases matched at \( x = 0 \). An end-effect term was then added, leaving the position of the closed end to be determined. Taylor used Fourier expansion methods; Defant used collocation and obtained the same results as Taylor. The limitation of these methods is that it is difficult to adapt them to the case when Poincaré mode propagation in the canal is possible.

In this paper, the origin of coordinates is chosen naturally, at one corner of the canal; the outgoing Kelvin wave is assigned an arbitrary amplitude and phase. The end effect is designed to exclude incoming Poincaré waves. Collocation is used to determine the relevant coefficients. The results are in agreement with those found by Taylor and Defant, for the case they considered.

2. Formulation of the Model. We consider a semi-infinite canal rotating about the vertical axis with an angular velocity of \( \frac{1}{2} \Omega \). The horizontal coordinates are \((x, y)\), defined such that the canal occupies the region \( x \geq 0, 0 \leq y \leq l \). The velocity components are \((u, v)\), and the canal depth is \( h \), assumed constant. Incident and reflected Kelvin waves are defined by

\[
\begin{align*}
\zeta_I &= e^{-i(\omega/c_0)(x-y)} + i(\omega t + \alpha x), \\
\zeta_R &= Re^{-i(\omega/c_0)y + i(\omega t - \alpha x)},
\end{align*}
\]

where \( \zeta, \omega, \alpha, \) and \( g \) measure the surface elevation, time, and gravity, respectively; \( \omega, \alpha, \) and \( c_0 \) are positive, and \( c_0^2 = gh/\alpha^2 \).

Associated velocities are

\[
\begin{align*}
u_I &= -(g/c_0)\zeta_I, & u_R &= (g/c_0)\zeta_R, & v_I &= v_R = 0
\end{align*}
\]

It is not possible, by choice of the complex constant \( R \), to construct a nodal line across the canal; an end effect, \( u_E, v_E, \zeta_E \), is required such that

\[
\begin{align*}
u(0, y) = & \ u(x, 0) = \ u(x, l) = 0
\end{align*}
\]

where \( u = u_E + u_I + u_R, \ v = v_E. \) This effect may be written as a set of Poincaré modes:

\[
u_E = (g/c_0) \sum_1^\infty \gamma_n \sin (r_n y) e^{-n x + i\omega t},
\]

where \( r_n = (n\pi/l) \), since these modes form a complete orthogonal set on \([0, l]\).

\[
\begin{align*}
s_n^2 &= r_n^2 - \alpha^2, & c_0^2 \alpha^2 &= \omega^2 - \Omega^2.
\end{align*}
\]

If \( s_n^2 < 0 \), write \( s_n = i\alpha_n, \alpha_n > 0 \), to ensure outward propagation; otherwise, \( s_n > 0 \).

2. Recently, Hendershott and Speranza (1971) have applied Taylor's method to the problem of reflection in the Adriatic Sea and in the Gulf of California.
The linearized momentum equations may be written
\[
\begin{align*}
\hbar \chi^2 u &= i \omega \zeta_x + f \zeta_y, \\
\hbar \chi^2 v &= -f \zeta_x + i \omega \zeta_y;
\end{align*}
\]
thus,
\[
\begin{align*}
u_E &= (g/\epsilon_0) \sum \frac{\gamma_n \{A_n \cos (r_n y) + B_n \sin (r_n y)\} e^{-s_n x + i \omega t}}{1} \left. \right|_{y=0, l}, \\
\zeta_E &= \sum \gamma_n \{C_n \cos (r_n y) + D_n \sin (r_n y)\} e^{-s_n x + i \omega t},
\end{align*}
\]
where
\[
\begin{align*}
\Delta A_n &= s_n r_n \epsilon_o^2, & \Delta B_n &= -i \omega f, & \Delta C_n &= i \omega r_n \epsilon_o, \\
\Delta D_n &= -f s_n \epsilon_o, & \text{and} & \Delta &= \omega^2 + s_n^2 \epsilon_o^2.
\end{align*}
\]
Using these in (4),
\[
Re^{-(\ell/\epsilon_o)} - e^{-(\ell/\epsilon_o)}(l-y) + \sum \frac{\gamma_n \{A_n \cos (r_n y) + B_n \sin (r_n y)\} e^{-s_n x + i \omega t}}{1} = 0
\]
for all \( y \in [0, l] \).

3. **The Collocation and Convergence of Solutions.** The method of "collocation" or "point-matching" is now introduced. Eq. (9) presents the problem of inverting a matrix of infinite order. However, if the following method is used, the problem is greatly simplified.

Eq. (9) is taken to hold at a finite set of points on \([0, l]\), with the series truncated to a finite number of terms. If only the first \( N \) terms of the series are included, there remain \( N + 1 \) constants to be determined in (9); thus, setting \( u(0, y) = 0 \) at \( N + 1 \) points on \([0, l]\) should provide a solution. The points used here are \( y_k = (k - 1) l/N \). The result is a matrix equation \( M \xi = 0 \), where \( \xi^T = (\gamma_1, \gamma_2, \ldots, \gamma_N, R, -1) \), from which \( \xi \) is rapidly determined.

The collocation method is one frequently used in elastic and electromagnetic waveguide problems, with varying success. Its main advantage is its simplicity. Laura (1970) has given a summary of recent applications of the method.

The validity of this solution can be tested by the accuracy of eq. (9) between the collocation points, by the convergence to a limiting solution for increasing \( N \), and, if there are propagating Poincaré modes, by the balancing of incident and reflected energy. The energy flux \(^3 \) of the propagating \( n \)-th Poincaré mode is
\[
F_n = \frac{1}{4} \frac{\alpha_n \epsilon_0}{\omega} \left| \gamma_n \right|^2 \left( \frac{1 - \left( \frac{\alpha_n \epsilon_0}{\omega} \right)^2}{\left( \frac{\alpha_n \epsilon_0}{\omega} \right)^2} \right)^{-1}.
\]

3. Defined by kinetic energy as measured in the rotating frame, and by gravitational potential energy. These two components are in the ratio \( (\omega^2 + f^2) : (\omega^2 - f^2) \). Note that, for \( \omega = f \), there is no potential energy; this is the situation of an inertial oscillation.
Table I. Convergence of $R$ with increasing $N$; Taylor example.

| $N$ | $R$ | $|R|$ |
|-----|-----|------|
| 4   | 0.74367 - 0.66855i | 1.00000 |
| 6   | 0.74149 - 0.67097i | 1.00000 |
| 8   | 0.74110 - 0.67140i | 1.00000 |
| 10  | 0.74098 - 0.67152i | 1.00000 |
| Est. limit | 0.7409 - 0.6716i | 1 |

Taylor and Defant considered the case $\zeta = 0.5(\pi/l), (f/c_0) = 0.7(\pi/l)$. The use of a depth of 74 m and a period of 12 hr, which Taylor (1921) used in his study of the Irish Sea, results in an equivalent canal width of 500.5 km, with a value of $f$ corresponding to a point at 54.46° N. These measurements will be referred to as “Taylor’s example”. The end of the canal was determined to be at $x_1 = 0.427(l/\pi)$, corresponding in this paper to

$$R = e^{-2ix_1} = 0.742 - 0.670i.$$ 

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Table I lists the computed values of $R$ for various $N$. Convergence is rapid, approximately inverse cubic in $N$. There are no propagating Poincaré modes and, as expected, $|R| \equiv 1$.

Also examined here is the case with a period of 6 hr, other parameters unchanged. From eq. (6), the $n$-th Poincaré mode will propagate if $s_n^2 < 0$, i.e., $r_n^2 < \kappa^2$, and if the period $P < P_n = 2\pi\{f^2 + c_0^2r_n^2\}^{-1/2}$. Here, $P_1 = 8.45$ hr, $P_2 = 4.87$ hr, and the first mode will propagate outward. Table II lists the computed values of $R$ and $\gamma_1$ and the excess of outgoing over incoming energy for various $N$. The energy excess decreases with $N$ as inverse power 2.08 while the amplitudes converge at a similar rate.

Figs. 1 and 2 demonstrate the behavior of the ratio $Q = |u|/\max |v|$, on $x = 0$. In each case, the imbalance diminishes rapidly with $N$.

It is worth noting that, although $P_1$ is always less than a pendulum day, canals are possible with $P_1$ greater than a tidal period; thus, perfect reflection of Kelvin-wave tides need not always occur.

4. The Reflection. The incident Kelvin wave is reflected at the first corner $(\alpha, l)$ along the side $x = 0$, and, since the period is less than $P_o = 2\pi/f = 1$,

Table II. Variation in $|R|$, $|\gamma_1|$, and the outgoing energy excess, $E^+$, with increasing $N$; Taylor example, except period 6 hr.

| $N$ | $E^+$ | $|R|$ | $|\gamma_1|$ |
|-----|------|------|--------|
| 4   | 0.14550 | 0.35294 | 0.57209 |
| 6   | 0.06169 | 0.35463 | 0.54777 |
| 8   | 0.03415 | 0.35495 | 0.53958 |
| 10  | 0.02170 | 0.35505 | 0.53584 |
| Est. limit | 0 | 0.3551 | 0.529 |
pendulum day, Poincaré waves are radiated from the corner. This process is repeated at the second corner (0,0). The two Poincaré wave patterns, with their subsequent side reflections, combine to give the required end effect.

If the period is greater than $P_1$, these patterns do not generate a traveling mode; after a short distance the amplitude of the outgoing Kelvin wave is restored to that of the incoming wave. If less than $P_1$, the first Poincaré mode propagates down the canal, and the Kelvin wave is not completely restored.

Figure 1. Variation in $q$ on $x = 0$, for Taylor's example; period 12 hr. Values of $N$ as shown.

Figure 2. Variation in $q$ on $x = 0$, for Taylor's example; period 6 hr. Values of $N$ as shown.

Figure 3. The variation in $|R|$ around $P_1$, for the case $l = 465$ km, $h = 74$ m, $53^\circ$N. $N = 5$. 
Figure 4 A–D. Range and phase lines for Taylor’s example \((l = 500.5 \text{ km}, 54.46^\circ \text{N.} \ P_t = 8.46 \text{ hr})\).
Phases for maximum amplitude are shown in tenths of a period. Range interval is 0.2. Oscillation periods as follows: A 12.0 hr, B 10.0 hr, C 9.0 hr, D 8.6 hr.

The first-mode dispersion relation is

\[ \omega^2 = f^2 + c_0^2 (\alpha^2 + r^2) \]

hence, for periods just below \(P_1\), say \(P = P_1 - \Delta, \alpha \ll \Delta \ll P_1\), the phase and group velocities are

\[ c_P = o(\Delta^{-1/2}), \quad c_G = o(\Delta^{1/2}). \]

The mode therefore transmits energy of the order \(\Delta^{1/2}\), and it is suspected that \(|R| = 1 - o(\Delta^{1/2})\); this is clearly demonstrated in Fig. 3. It is also clear

Table III. Variation in \(R\) for periods around \(P_1\), for canal of width 465 km, depth 74 m, at 53°N. \(N = 5\).

| \(P\) (hr) | \(R\) | \(|R|\) |
|---|---|---|
| 8.070 | -0.61253 + 0.28461i | 0.67542 |
| 8.075 | -0.65841 + 0.30133i | 0.72409 |
| 8.080 | -0.72087 + 0.32503i | 0.79076 |
| 8.085 | -0.84237 + 0.37426i | 0.92177 |
| \(P_1 = 8.087\) | -0.912 + 0.403i | 1 |
| 8.090 | -0.97809 + 0.20819i | 1.00000 |
| 8.095 | -0.99395 + 0.10986i | 1.00000 |
| 8.100 | -0.99939 + 0.03481i | 1.00000 |
| 8.105 | -0.99960 - 0.02825i | 1.00000 |
that Poincaré mode propagation rapidly asserts itself as the dominant reflection mechanism.4

Similarly, for \( P = P_1 + \Delta, s_1 = o(\Delta^{1/2}) \), so that the length scale of the end effect is \( o(\Delta^{-1/2}) \); the phase of \( R \) undergoes a rapid change just above \( P = P_1 \), but little change just below. The values of \( R \) around \( P = P_1 \) are given in Table III.

The surface-elevation behavior for several cases around \( P = P_1 \) is given in Figs. 4A–4H. Note that the outgoing first Poincaré mode is concentrated toward the same side as the incoming Kelvin wave; this results from the condition \( v = 0, \) or \( fu + (g)(\partial \xi / \partial y) = 0, \) at the sides \( y = 0, y = 1. \) This effect is much less pronounced for the higher modes.

5. Conclusions. This study shows that the propagation of Poincaré waves in continental-shelf seas could be very prominent in tidal records. For example, the Gulf of Carpentaria (circa 140°E, 15°S) has a width of roughly 500 km, a depth of 50 m, and a pendulum day of about 50 hr. \( P_1 \) is close to 12 hr, and it is possible that any tidal Kelvin wave entering the Gulf would be entirely

4. In Fig. 3 and Table III, a width of 465 km and a latitude of 53° are used. These are as given in Defant (1961), later found to be incorrect. However, for Fig. 3 and Table III, only the general behavior was required, so no change was made.
reflected as the first Poincaré mode. Thus, techniques allowing for inclusion of Poincaré propagation can be quite useful.

It is also clear that collocation is an efficient method of calculation for the canal geometry and that it should be useful in slightly more complicated geometries. Further investigations of the method are currently under way.

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**APPENDIX: NOTES ON TWO LIMITS**

Two limiting cases of interest are those of a narrow canal and of a short-period oscillation; it is expected that the effects of rotation should be small.

Define \( F = f/\omega, \varepsilon = \omega l/c_0 = \alpha l; \) then \( F \) is the period in pendulum days, \( 2\pi\varepsilon \) is the ratio of canal width to wavelength. Putting \( x = Xl, y = Yl, \) and \( \omega t = T, \) the various terms are

\[
\zeta_I = e^{-\varepsilon F(1-Y)+i(T+\varepsilon X)} \\
\zeta_R = Re^{-\varepsilon FY+i(T-\varepsilon X)}, \\
\xi_E = \sum_{n=1}^{\infty} \gamma_n \{A_n \cos(n\pi Y) + B_n \sin(n\pi Y)\} e^{-S_n X+iT}, \\
v_E = (g/c_0) \sum_{n=1}^{\infty} \gamma_n \sin(n\pi Y) e^{-S_n X+iT}, \\
\zeta_E = \sum_{n=1}^{\infty} \gamma_n \{C_n \cos(n\pi Y) + D_n \sin(n\pi Y)\} e^{-S_n X+iT},
\]

(A1) (A2) (A3)
where
\[ S_n^2 = n^2 \pi^2 - \varepsilon^2 (1 - F^2), \quad \Lambda = n^2 \pi^2 + \varepsilon^2 F^2, \]
\[ \Delta A_n = n \pi S_n, \quad \Delta B_n = -i F \varepsilon^2, \quad \Delta C_n = i n \pi \varepsilon, \quad \Delta D_n = -\varepsilon F S_n. \]

(i) Narrow canal: \( \varepsilon \ll 1 \).

\[ A_n = 1 + o(\varepsilon^2), \quad B_n = o(\varepsilon^2), \]
\[ C_n = \left( \frac{i}{n \pi} \right) \varepsilon + o(\varepsilon^3), \quad D_n = -\left( \frac{F}{n \pi} \right) \varepsilon + o(\varepsilon^3), \]
and
\[ u(0, y) = o = \left( g/c_0 \right) \left\{ \sum_{1}^{\infty} \gamma_n \cos (n \pi Y) - i \varepsilon F (1 - Y) + \right\} \]
\[ + R (1 - \varepsilon F Y) + o(\varepsilon^2) \} e^{i T}. \] (A4)

Integrating over \( 0 \leq Y \leq 1 \),
\[ R = 1 + o(\varepsilon^2), \] (A5)
and so
\[ \sum_{1}^{\infty} \gamma_n \cos (n \pi Y) = \varepsilon F (2 Y - 1) + o(\varepsilon^2), \]
from which
\[ \gamma_k = \frac{4 \varepsilon F}{k^2 \pi^2} \left[ (-1)^k - 1 \right] + o(\varepsilon^2). \] (A6)

Accordingly, \( \zeta_E = o(\varepsilon^2) \), and the end effect is virtually absent.

From (A1) and (A3), using (A5) and (A6),
\[ \zeta = \{(2 - \varepsilon F) \cos (\varepsilon X) + i(\varepsilon F)(2Y - 1) \sin (\varepsilon X)\} e^{iT} + o(\varepsilon^2). \] (A7)

The tidal behavior approximates that expected without rotation.

(ii) Short period: \( F \ll 1 \).

Let \( \sigma_n^2 = n^2 \pi^2 - \varepsilon^2 \). Then
\[ A_n = \left( \frac{\sigma_n}{n \pi} \right) + o(F^2), \quad B_n = -\left( \frac{i \varepsilon^2}{n^2 \pi^2} \right) F + o(F^3), \]
\[ C_n = \left( \frac{i \varepsilon}{n \pi} \right) + o(F^2), \quad D_n = -\left( \frac{\varepsilon \sigma_n}{n \pi} \right) F + o(F^3), \]
and
\[ u(0, y) = o = \left( g/c_0 \right) \left\{ \sum_{1}^{\infty} \gamma_n \left[ \left( \frac{\sigma_n}{n \pi} \right) \cos (n \pi Y) - \frac{i \varepsilon^2}{n^2 \pi^2} F \sin (n \pi Y) \right] - \right\} \]
\[ - 1 + \varepsilon F (1 - Y) + R (1 - \varepsilon F Y) + o(F^2) \} e^{iT}. \] (A8)
Table IV. Behavior of $R$ and $(\gamma)$ for different values of $F$, $\varepsilon$. Case $N = 6$.

<table>
<thead>
<tr>
<th>$I$</th>
<th>500.5 km</th>
<th>10.01 km</th>
<th>25.03 km</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>12 hr</td>
<td>12 hr</td>
<td>36 min</td>
</tr>
<tr>
<td>$F$</td>
<td>0.814</td>
<td>0.814</td>
<td>0.041</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>2.70</td>
<td>0.054</td>
<td>2.70</td>
</tr>
<tr>
<td>$R$</td>
<td>$.74149 - .67097i$</td>
<td>$1.00000 - .00001i$</td>
<td>$.99999 - .00336i$</td>
</tr>
</tbody>
</table>

Collocation Equation (A6)


gn \cdot \frac{i \varepsilon^2 F}{n^3 \pi^3} \{( - 1)^n - 1 \} = (1 - R)(1 -^{1/2} \varepsilon F) + o(F^2),

so

$$R = 1 + o(F).$$ (A9)

Therefore, $\gamma_n$ and $\zeta_B$ are both $o(F)$; the end-effect tide and the Kelvin-wave height variation are of similar magnitude:

$$\zeta = 2 \cos(\varepsilon X)e^{iT} + o(F).$$ (A10)

Table IV presents the solution coefficients for the Taylor example and for two other examples with the values of $\varepsilon$ and $F$ reduced by factors of 50 and 20, respectively. Note, in particular, the excellent agreement for $\varepsilon \ll 1$. The estimated values by eq. (A6) are also listed in this case. Here, in fact, it is suggested that higher-order analysis may show $R = 1 + o(\varepsilon^4)$. The divergence for higher values of $n$ is almost certainly due to having taken $N = 6$ in this example.