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The Electromagnetic Field of Long and Intermediate Water Waves

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ABSTRACT

The electromagnetic field induced by long and intermediate water waves (e.g., tsunami) is dependent on the time variation and spatial variation of the water motion and on the earth's electrical conductivity structure. The water motion is approximated by a plane progressive wave on a nonrotating flat infinite ocean of uniform depth. The earth's conductivity is approximated by plane uniformly conductive layers that represent the ocean, the bottom sediments, and the highly conductive mantle. The analytical solution to the electromagnetic problem shows the important influence of frequency, wave number, and oceanic and mantle conductivity, all of which modify the field through both self induction and mutual induction. Amplitudes of the electromagnetic field are tabulated for a range of tsunami frequencies.

1. Introduction. Electric and magnetic fields due to water waves are caused by the emfs that result from the oscillatory motion of the conducting seawater in the geomagnetic field. The field for long and intermediate water waves (e.g., tsunami) depends not only on the time and spatial variation of the water motion but also on the earth's electrical conductivity structure. In this paper the water motion is approximated by a plane progressive wave on a flat ocean of uniform depth. The principal conducting elements of the earth are: the ocean, the water-saturated bottom sediments, and the highly conductive mantle at depth. These are approximated by plane horizontal uniformly conductive layers. The electromagnetic problem then has a simple analytical solution that is useful as a first step in estimating the magnitude of the induced electromagnetic field and in displaying the role of frequency, wave number, geomagnetic field, and the various conductive elements of the earth. In addition, the formulae could be used to estimate wave motion from electromagnetic field observations.

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The present study extends, to long and intermediate waves, Weaver's (1965) theoretical study of magnetic fields induced by short waves, such as swell and sea waves.

2. Hydrodynamic Theory. It is essential, before specifying the electromagnetic problem, that the model of the wave adopted be consistent with the hydrodynamic equations. Here the wave motion is limited (i) to frequencies, \( \omega \), that are high enough so that the rotational effects of the earth are minor \( (\omega \gg \Omega_E) \), \( \Omega_E = 7.3 \times 10^{-5} \text{ sec}^{-1} \), earth's rotation rate) and (ii) to horizontal wave numbers, \( k \), that are large enough so that the wave is not modified by the earth's curvature \( (k \gg 1/R_E) \), \( R_E = 6.4 \times 10^6 \text{ m} \), earth's radius). Furthermore, the effect of continental borders and bottom topography are not considered. Thus the ocean is idealized as a flat nonrotating ocean of infinite extent and of uniform depth, \( h \). Let the coordinate system \((x, y, z)\) be rectangular, with \( z \) measured up from the ocean bottom and with \( x \) of arbitrary direction.

The appropriate plane wave is an irrotational wave of small amplitude in an incompressible inviscid fluid. The hydrodynamic equations, in terms of a velocity potential, \( \phi \), reduce to Laplace's equation (e.g., Coulson 1958)

\[
\nabla^2 \phi = 0. \tag{1}
\]

The boundary conditions on the free surface, \( z = h \), are

\[
\frac{\partial \phi}{\partial t} + g \frac{\partial \zeta}{\partial t} = 0, \quad \frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial z} = 0; \tag{2}
\]
on the bottom, \( z = 0 \), they are

\[
\frac{\partial \phi}{\partial z} = 0. \tag{3}
\]

In these equations the particle velocity is \( \mathbf{U} = \nabla \phi \); \( g = 9.8 \text{ (m/sec}^2 \) is the acceleration of gravity; \( \zeta \) is the displacement of the free surface. The solution for a progressive wave,

\[
\zeta = A \exp [i(kx - \omega t)], \tag{4}
\]
satisfying (1) – (3), is

\[
\phi = -i \frac{g}{\omega} [\cosh (kz)/\cosh (kh)] \zeta. \tag{5}
\]
The frequency and wave number are related by the propagation equation

\[
\omega^2 = gk \tanh (kh). \tag{6}
\]
The components of the particle velocity are

\[
U_x = \frac{\omega}{\sinh (kh)} \cosh (kz) \zeta, \quad U_y = 0, \quad U_z = -i \tanh (kz) U_x. \tag{7}
\]

3. Electrical Conductivity. Previous investigations (see, for example, Chapman et al. 1940, Chapman 1956, Cantwell et al. 1960, Banks 1969, Bullard
et al., in press, Cox et al., in press) indicate that the principal conducting elements of the earth are: an ionosphere about 100 km above the earth's surface with a conductivity of $\sigma_i < 10^{-3}$ ohm$^{-1}$ m$^{-1}$ and a thickness of $h_i \sim 50$ km; an ocean, $\sigma \approx 3.3$ ohm$^{-1}$ m$^{-1}$ and $h \approx 5$ km, with a layer of conducting sediments, $\sigma_s \lesssim 0.5$ $\sigma$ and $h_s \lesssim 0.5$ km; a conductive mantle at a depth of a few tens to hundreds of kilometers.

The conductive mantle beneath the Pacific Ocean at 600 km from California is shallower (depth $H_m \sim 30$ km) and more conductive ($\sigma_m \sim 0.4$ ohm$^{-1}$ m$^{-1}$) than the mantle beneath the adjacent continent ($H_m \sim 80$ km and $\sigma_m \sim 0.03$ ohm$^{-1}$ m$^{-1}$) (Cox et al., in press). These values, which were inferred from electromagnetic signals with $\omega$ of the order of $10^{-3}$ sec$^{-1}$, are appropriate for the present study and are referred to in this paper as the shallow-mantle and deep-mantle models, respectively. The deep-mantle model is consistent with the mantle conductivity beneath Massachusetts (Cantwell et al. 1960). At depths greater than 400 km, the conductivity rises to 1 ohm$^{-1}$ m$^{-1}$ (Banks 1969). For a recent discussion of deep-mantle conductivity, see the review by Price (1970). However, this deeper and more conductive part of the mantle can be ignored, because signals with $\omega \gg 10^{-4}$ sec$^{-1}$ are completely attenuated at such depths. It has been proposed by Caner and Auld (1968) that the conductive mantle at depths of the order of 100 km beneath Victoria, British Columbia, is separated, by a 500-km zone of low conductivity, from the deeper and more conductive mantle. This conclusion was based on an interpretation of their magnetotelluric data by use of a model consisting of horizontal layers of uniform conductivity. However, they observed that the telluric field was strongly north-south polarized at all frequencies. This suggests an important shallow east-west conductivity structure (perhaps the Straits of Juan de Fuca) that should have been included in the model. Therefore their proposed zone of low conductivity is not considered here. Also omitted from consideration are: (i) the conductivity of crustal and mantle material above the conductive mantle because its conductivity probably does not exceed $10^{-3}$ ohm$^{-1}$ m$^{-1}$ on the average (Watt et al. 1963); (ii) the ionosphere because its conductivity is so much less than that of the ocean; and (iii) spatial variations in the conductivity within the ocean and in the mantle.

The earth's conductivity structure is thus idealized to horizontal uniformly conductive layers (Fig. 1), with the conductive mantle approximated by an infinite half space of uniform conductivity. The above estimates of conductivity and depths are used for determining the magnitudes of the fields.
4. Electromagnetic Theory. The basic equations are Maxwell's equations and Ohm's law for a moving conductor. Displacement currents may be neglected because the phase speed, $\omega/k$, of the source is vanishingly small when compared with the speed of light, $c$. Maxwell's equations (MKS units) are:

$$\nabla \times B = \mu J, \quad \nabla \times E = -\partial B/\partial t, \quad \nabla \cdot B = 0, \quad \nabla \cdot E = \varrho/\varepsilon. \quad (8)$$

In these equations, $J$ is the electric current (amp/m$^2$), $E$ is the in-situ electric field (volt/m), $B$ is the magnetic field (weber/m$^3$), $\varrho$ is the electric-charge density (coul/m$^3$), $\mu = 4\pi \times 10^{-7}$ (ohm sec/m) is the magnetic permeability, assumed everywhere equal to that in vacuo, and $\varepsilon = 1/(\mu c^2)$ (ohm$^{-1}$ m$^{-1}$ sec) is the electric permittivity.

The induced field, $B$, is vanishingly small when compared with the steady geomagnetic field, $F$. Hence Ohm's law for a conductor moving with fluid velocity, $U$, is

$$J = \sigma (E + U \times F), \quad (9)$$

from which the nonlinear term, $U \times B$, has been dropped. The electric field, $E'$, seen by an observer moving with the fluid velocity, is

$$E' = E + U \times F = J/\sigma \quad (10)$$

and is referred to as the GEK (geomagnetic electrokinetograph) field.

The spatial variations of $F$, compared with $U$, can be ignored; i.e., $(U \cdot \nabla) F \ll (F \cdot \nabla) U$. Then elimination of $E$ and $J$ from (8)–(9) together with $\nabla \cdot F = 0$ and $\nabla \cdot U = 0$ yields the differential equation

$$\nabla^2 B = \mu \sigma \partial B/\partial t - \mu \sigma (F \cdot \nabla) U. \quad (11)$$

The boundary condition is continuity of $B$ across a finite discontinuity in conductivity. If one of the conductors is infinitely conducting, only the normal component of $B$ is continuous, and it vanishes at the interface.

Eqs. (8) imply that $\nabla \cdot J = 0$; thus, if one can ignore displacement currents, then one can also ignore, to the same order, the term $\partial \varrho/\partial t$ in the electric-current continuity equation (Backus 1958). Hence, within a uniform conductor, $\nabla \cdot E' = 0$. Then $\nabla \cdot E = -F \cdot \nabla \times U = 0$. This follows because $U$ is irrotational and because the geomagnetic field is not caused by electric currents in the sea; i.e., $\nabla \times F = 0$. Thus electric charges do not accumulate within the uniform conductor.

By Stokes' theorem, the line integral of $U \times F$, the emfs, about a loop in the $x,z$ plane, for an incompressible fluid motion and a uniform geomagnetic field, is

$$\oint (U \times F) \cdot ds = \int \int (F \cdot \nabla) U_y \, dx \, dz. \quad (12)$$
Since $U_y = 0$ from (7), the net emfs about any $x, z$ loop vanishes. Then, since the oceanic conductivity is assumed uniform, $\mathbf{J}_x = \mathbf{J}_z = 0$ and, consequently, $B_y = 0$.

5. The Induced Electromagnetic Fields. The water motion adopted has the form

$$U = u(z) \exp [i(kx - \omega t)].$$

The conductivity model consists of only horizontal layers of uniform conductivity. Therefore, the induced electromagnetic field can also be expected to be of the form

$$\begin{pmatrix} B \\ E \\ J \end{pmatrix} = \begin{pmatrix} b(z) \\ e(z) \\ j(z) \end{pmatrix} \exp [i(kx - \omega t)].$$

Substitution of (13) into (8) – (11) gives

$$\begin{align*}
    b_x &= (i/k) db_z/dz, \quad b_y = 0, \\
    e_x &= u_z F_y, \quad e_y = (\omega/k) b_z, \quad e_z = -u_x F_y, \\
    e'_x &= e'_z = 0, \quad j_x = j_z = 0, \\
    e'_y &= j_y/\sigma = (\omega/k) b_z - u_x F_z + u_z F_x.
\end{align*}$$

All components can be computed from $b_z(z)$ and $u(z)$, and the electromagnetic problem reduces to solving the vertical component of (11). Within the ocean it is

$$d^2 b_z/dz^2 = (k^2 - i \mu \omega) b_z + i \mu \sigma k (u_x F_z - u_z F_x);$$

outside the ocean it is

$$d^2 b_z/dz^2 = (k^2 - i \mu \omega) b_z.$$ 

The boundary conditions are continuity of $b_z$ and $db_z/dz$ at the interfaces $z = h$, $0$, $-h$, $-H_m$.

Eq. (16) shows that a sufficient condition for ignoring the conductivity of the ionosphere and crust is $k^2 \gg \mu \omega$. This condition is satisfied if $\omega \gg 5 \times 10^{-5} \text{ sec}^{-1}$ for waves consistent with propagation equation (6) and for an ionosphere or crust with a conductivity of $10^{-3} \text{ ohm}^{-1} \text{ m}^{-1}$. Hence, the conductivities of the ionosphere and crust are ignored.

The differential equation (16) and the homogeneous part of (15) have constant coefficients and therefore have exponential solutions of the form $\exp [\pm (k^2 - i \mu \omega)^{1/2}z]$, with $\sigma$ varying as in Fig. 1. The particular solution of (15) has the form $\exp (\pm kz)$. The complete solution can be developed in terms of these exponentials by multiplying them by unknown coefficients.
that are evaluated by making the solution and its first derivative continuous at the various interfaces.

To generalize the result, the solution is expressed in dimensionless form. Suitable nondimensional parameters are: 

\[ \hat{A} = \frac{A}{h}; \quad \hat{z} = \frac{z}{h}; \quad \hat{\omega} = \omega \left(\frac{h}{g}\right)^{1/2}; \]

\[ \hat{k} = \frac{k}{h}; \quad \hat{h}_s = \frac{h_s}{h}; \quad \hat{h}_m = \left(\frac{H_m - h_s}{h}\right); \quad \hat{\sigma}_j = \mu \sigma_j h \left(\frac{g h}{h_s}\right)^{1/2} \]

\[ \lambda_j^2 = \hat{k}^2 - i \hat{\omega} \hat{\sigma}_j \]

for \( \hat{\sigma}_j = \hat{\sigma}_s, \hat{\sigma}_m. \) The propagation equation (6) is \( \hat{\omega}^2 = \hat{k} \tanh(\hat{k}). \) The dimensionless fields are: 

\[ \hat{F} = \frac{F}{F}; \quad \hat{B} = \frac{b}{(\hat{A}F)}; \quad \hat{E} = \frac{e}{(\hat{A}F \omega / k)}; \quad \hat{J} = \frac{j(\sigma \hat{A}F \omega / k)}{F}; \]

here \( F \) is the magnitude of \( F. \) The solution for the field within the ocean is

\[
\hat{B}_z = \left[\frac{\hat{k}}{\sinh(\hat{k})}\right]\left[\hat{F}_z \{\cosh(\hat{k}\hat{z}) - \alpha_\times \hat{X}_1(\hat{z}) - \hat{X}_2(\hat{z})\} + i\hat{F}_x \{\sinh(\hat{k}\hat{z}) + \hat{X}_1(\hat{z}) - \hat{X}_2(\hat{z})\}\right]
\]

where

\[
\hat{X}_1 = (1 + \alpha)^{-1} \left[\cosh(\alpha) - \hat{k} \alpha^{-1} \alpha \sinh(\alpha)\right]\left[\cosh\{\alpha(1 - \hat{z})\} + \hat{k} \alpha^{-1} \sinh\{\alpha(1 - \hat{z})\}\right],
\]

\[
\hat{X}_2 = (1 + \alpha)^{-1} \exp(\hat{k})\left[\cosh\{\alpha(1 - \hat{z})\} - \hat{k} \alpha^{-1} \alpha \sinh\{\alpha(1 - \hat{z})\}\right],
\]

and

\[
\alpha = \alpha / \hat{k} \left[\alpha + \hat{k} \alpha_s \coth(\alpha) - \hat{\alpha}_s \coth(\alpha) \hat{k} \alpha_s\right]^{-1}
\]

\[
\alpha_s = \left(\alpha_s / \hat{k}\right) \left[\alpha_s + \hat{k} \alpha_m \coth(\alpha_s \hat{h}_s)\right] \left[\alpha_s \coth(\alpha_s \hat{h}_s) + \hat{k} \alpha_m\right]^{-1}
\]

\[\alpha_m = [\hat{k} + \alpha_m \coth(\hat{k}\hat{h}_m)\] \[\hat{k} \coth(\hat{k}\hat{h}_m) + \alpha_m\]^{-1}.

The \( \alpha \)'s defined above are proportional, respectively, to the surface admittances (Wait 1962: 17) associated with the interfaces of the ocean/bottom sediments, the bottom sediments/nonconducting mantle, and the nonconducting/conducting mantle. The electric-current density and GEK field are

\[
\hat{J}_y = \hat{E}_y' = \hat{B}_z - \left[\frac{\hat{k}}{\sinh(\hat{k})}\right]\left[\hat{F}_z \cosh(\hat{k}\hat{z}) + i\hat{F}_x \sinh(\hat{k}\hat{z})\right].
\]

The horizontal magnetic field on the sea floor, \( \hat{z} = 0 \), is

\[
\hat{B}_{z}^- = i\alpha_s \hat{B}_z;
\]

at the sea surface, \( \hat{z} = 1 \), it is

\[
\hat{B}_{z}^+ = -i\hat{B}_z^+.
\]

The magnetic field in the air, \( \hat{z} > 1 \), is

\[
\hat{B}_{z}(\text{air}) = -i\hat{B}_z(\text{air}) = -i\hat{B}_z^+ \exp[-\hat{k}(\hat{z} - 1)];
\]

this decreases exponentially upward from the sea surface. The electric field within the ocean is

\[
\hat{E}_x = -i[k/sinh(k)]\hat{F}_{y} \sinh(k\hat{z}), \quad \hat{E}_y = \hat{B}_z, \quad \hat{E}_z = -i \coth(k\hat{z}) \hat{E}_x.
\]
On the sea floor, \( \hat{E}_x = 0 \). The magnetic field recorded by a total-field magnetometer is

\[
\hat{B}_T = \hat{F}_x \hat{B}_x + \hat{F}_z \hat{B}_z,
\]

and the bottom or surface values can be found by using (20) or (21), respectively.

In Fig. 2, the amplitudes of the above dimensionless electromagnetic field are plotted as a function of frequency for the case of a deep ocean \((h = 5000 \text{ m})\) with conducting sediments \((h_s = 500 \text{ m})\). The curves show that, for low frequencies of less than about 10 cph, the horizontal geomagnetic field, \(F_x\), can be ignored because \(U_z \ll U_x\); but the conductive mantle can not be ignored because of mutual-induction coupling with the mantle. The conductive mantle, especially if it is shallow, enhances the horizontal magnetic field beneath the ocean relative to its value on the sea surface. This is due to the
strong electric currents induced in the highly conductive mantle. At higher frequencies, greater than about 50 cph, the conductivity of the mantle can be ignored, but the effect of the horizontal and vertical geomagnetic fields is comparable because $U_z \sim U_x$. The surface field is now larger than the field on the sea floor because the mantle electric currents are weak and the wave motion is largest at the sea surface.

In Fig. 3, for comparative purposes, the nondimensional amplitudes for a shallow shelf-type ocean ($h = 200$ m) with conductive sediments ($h_s = 500$ m) are plotted as a function of frequency. Here one sees that the effect of the horizontal geomagnetic field can be ignored for frequencies of less than 100 cph. Furthermore, the electric and magnetic fields are much smaller than those in the deep-ocean case and much less dependent on the conductive mantle. This follows because self induction and mutual induction are very small (§ 6).

The fields for the above deep-ocean and shallow-ocean case were also computed, ignoring the conductive-bottom sediments. In Figs. 2 and 3, the curves in all cases with bottom sediments are barely distinguishable from the curves without bottom sediments.
Finally, the dimensional field values can be determined from Figs. 2 and 3 by assigning values to the wave amplitude, $A$, and to the geomagnetic field, $F_x$ and $F_z$.

6. Superconductive Mantle. It is sometimes appropriate, and usually simpler, to replace the finite conductive mantle with a superconductive mantle at an apparent depth, $z = -H_a$. The surface-admittance term then becomes the real quantity

$$\alpha_a = \coth (\hat{k}h_a),$$

where $\hat{h}_a = (H_a - h_a)/h$. The solution for this case can be found by merely replacing the complex quantity $\alpha_m$ in (18) with $\alpha_a$. However, the effect of the mantle becomes really important only when $\alpha_m$ is much larger than unity, i.e., for very long waves for which $\hat{k}h_m << 1$ and $\hat{\omega} << \hat{\sigma}_m$. Equating $\alpha_a$ with $\alpha_m$, with $\alpha_m$ replaced by its modulus, $|\alpha_m|$, one obtains for the above approximations the following expression:

$$\hat{h}_a = \hat{h}_m + (\hat{\sigma}_m\hat{\omega})^{-1/2}.$$  (26)

The last term on the right is proportional to the skin depth within the finite conductive mantle. Thus the field penetrates to greater depths as the frequency is decreased, and the superconductive mantle must, accordingly, be placed deeper. The equivalence between solutions for a finite and infinitely conductive mantle can only be crude because phase shifts are ignored. However, (26) is useful because it applies to any long-wave electromagnetic signal. For example, Mason (1963), in a study of ionospheric-induced magnetic variations at Christmas Island, interpreted his results in terms of a superconductive mantle. His results, using (26), show that the mantle beneath Christmas Island is nearly equivalent to that in the deep-mantle model used here; the rough estimates are: $H_m \sim 100$ km and $\sigma_m \sim 0.1$ ohm$^{-1}$m$^{-1}$.

7. Self Induction and Mutual Induction. The short-wave problem, $\hat{k} >> 1$, has been studied by Young et al. (1920), Longuet-Higgins et al. (1954), Crews and Futterman (1962), Warburton and Caminiti (1964), Maclure et al. (1964), Fraser (1965, 1966), and Weaver (1965). In contrast to the work of Weaver, the solution obtained by the other workers assumed from the start that $\partial B/\partial t = 0$. The consequence of this assumption is now examined.

Consider the self induction associated with a simple circuit consisting of a resistor, $R$, in series with an inductor, $L$, and an emf, $V$, proportional to \( \exp(-i\omega t) \). The electric current $I$ satisfies the equation

$$(1 - iQ)I = V/R,$$  (27)
where \( Q = \omega L/R \) is a real quantity and is related to the ratio of the energy stored to the energy dissipated per cycle. If the time rate of change of the magnetic flux through the inductor vanishes, \( Q = 0 \). The electric current is then \( I_{Q=0} = V/R \) and

\[
Q = i(I_{Q=0} - 1)/I. \tag{28}
\]

Returning to the wave problem, let \( E = -\nabla \varphi - \partial A/\partial t \), where \( \varphi \) is the electrostatic potential and \( A \) is the magnetic vector potential, with \( B = \nabla \times A \). Then, by (9), \( J = \sigma(U \times F) - \sigma \nabla \varphi - \sigma \partial A/\partial t \). In a horizontally unbounded ocean of uniform depth and conductivity, there is no horizontal electrostatic field; i.e., there are no free charges within the ocean (§ 4) and no vertical boundaries on which charges can accumulate. Furthermore, \( \mathcal{J}_x = \mathcal{J}_z = 0 \). Thus \( \mathcal{J}_y = \sigma(U \times F)_y - \sigma \partial A_y/\partial t \). Letting \( \partial B/\partial t = 0 \) amounts to letting \( \partial A/\partial t = 0 \). Also, \( B_z = ikA_y \). Thus, by (28),

\[
Q = i\sigma(\partial A_y/\partial t)/\mathcal{J}_y = -iB_z/\mathcal{J}_y, \tag{29}
\]

which, in the present paper, is a function of the vertical coordinate, \( \hat{z} \). Hence, ignoring magnetic time variations is equivalent to ignoring self induction and mutual induction.

Mutual induction comes about because there are other conductive regions besides the ocean. The exact equivalent circuit for the present wave problem consists of two additional R-L circuits—one for the sediments and one for the conductive mantle. These are coupled to the ocean circuit and to each other by mutual induction. Here it is useful to retain \( Q \) as defined by (29), but it must be complex so as to include mutual induction. \( Q \) is referred to here as the induction parameter; its magnitude, relative to unity, is a measure of its importance.

In Fig. 4 the amplitude of the induction parameter is plotted as a function of frequency for the case \( F_x = 0 \). Case \( F_z = 0 \) gives equivalent results. The three upper curves, for a deep ocean (\( h = 5000 \) m), show that self induction and mutual induction are especially important in the tsunami frequency range (1–10 cph). The curve for no mantle shows, for frequencies of less than about 10 cph, that mutual induction of the conductive mantle reduces the induction parameter. Above 10 cph, mutual induction can be ignored, and above 100 cph—i.e., in the short-wave region—both self induction and mutual induction can be ignored. The bottom curve shows that self induction and mutual induction can be completely ignored for a shallow ocean (\( h = 200 \) m). Hence, the induction parameter depends critically on the oceanic depth.

8. Short-wave and Long-wave Approximation. There are two useful special cases: (i) \( \hat{\omega} \gg \hat{\sigma} \), and (ii) \( \hat{\omega} \ll \hat{\sigma} \). For case (i), the approximation implies that the wave period is large compared with the free decay time of the magnetic field associated with the oceanic electric currents. Hence this case corresponds to large dissipation of the field within the ocean. For case (ii), the wave period
Figure 4. The amplitude of the induction parameter, $Q$, at the sea surface as a function of frequency, $f = \omega/(2\pi)$, for the case $F_x = 0$. The bottom curve corresponds to a shallow ocean ($h = 200$ m, $\sigma = 3.3$ ohm$^{-1}$ m$^{-1}$), the upper three curves to a deep ocean ($h = 5000$ m, $\sigma = 3.3$ ohm$^{-1}$ m$^{-1}$). The curves corresponding to a shallow mantle ($H_m = 30$ km, $\sigma_m = 0.4$ ohm$^{-1}$ m$^{-1}$), a deep mantle ($H_m = 80$ km, $\sigma_m = 0.03$ ohm$^{-1}$ m$^{-1}$), and no mantle ($\sigma_m = 0$) are noted.

is small compared with the free decay time of the magnetic field associated with the oceanic electric currents. Hence this case corresponds to small dissipation within the ocean, and the field can therefore penetrate into the mantle. For the deep ocean, the conductivity is $\hat{\sigma} \approx 5 = o(1)$ $^2$ Hence case (i) also implies that $\hat{\omega} \gg o(1)$, which in turn implies short waves. Case (ii) implies that $\hat{\omega} \ll o(1)$, which in turn implies long waves.

(i) SHORT-WAVE CASE. Short waves, $\hat{k} \gg 1$, have been treated by Weaver (1965) and others and will not be considered here in detail. However, in the general case for any wave number, $\text{tanh}(\hat{k}) < \hat{k}$, so that, from the propagation equation, $\hat{\omega}^2 < \hat{k}^2$. Then for $\hat{\omega} \gg \hat{\sigma}$, one has $\hat{\nu} = \hat{k} + o(\hat{\sigma}/\hat{\omega})$; the fields, from (17)-(23), are:

$$
\begin{align*}
\hat{E}_y &= \hat{B}_z = o(\hat{\sigma}/\hat{\omega}), \\
\hat{B}_x &= o(\hat{\sigma}/\hat{\omega}), \\
\hat{f}_y &= \hat{E}_y = -[\hat{k}/\sinh(\hat{k})][\hat{F}_z \cosh(\hat{k}z) + i\hat{F}_x \sinh(\hat{k}z)] + o(\hat{\sigma}/\hat{\omega}), \\
\hat{E}_x &= - i[\hat{k}/\sinh(\hat{k})] \hat{F}_y \sinh(\hat{k}z), \\
\hat{E}_z &= - i \coth(\hat{k}z) \hat{E}_x.
\end{align*}
$$

$^2$ $g = o[f(e)]$ means $g/f(e)$ is bounded as $e \rightarrow 0$. 


Note that the approximate electric current is \( J_y = \sigma(U \times F)_y \), which implies that self and mutual induction can be ignored. To the next order, the magnetic field on the sea surface for short waves, \( \hat{k} \gg 1 \), is
\[
\hat{B}_x^+ = -i^{1/4}(\hat{\sigma}/\omega)(\hat{F}_x + i\hat{F}_x) + O(\hat{\sigma}^2/\omega^2);
\]
this is plotted as a function of frequency in Figs. 2 and 3 (short dashes). These curves show that the short-wave approximation approaches the solution as \( \omega \to \infty \). The induction parameter is
\[
Q = i^{1/4}\hat{\sigma}/\omega^3[1 + O(\hat{\sigma}/\omega)].
\]

Thus, for short waves the electric current is confined to a region of depth \( 1/k \) near the sea surface and flows in bands along the wave crests in one direction and along the wave troughs in the opposite direction. These alternate bands of electric current are not modified much by self induction and mutual induction. Therefore the electric current follows the wave motion, i.e., \( J_y = \sigma(U \times F)_y \), and the magnetic field can be determined by the Biot-Savart Law—i.e., by a volume integration over the electric currents.

(ii) Long-wave Case. The propagation equation for long waves, \( \hat{k} \ll 1 \), is
\[
\hat{\omega} = \hat{k} + O(\hat{k}^3).
\]
Then, for \( \hat{\omega} \ll \hat{\sigma} \), one has \( \kappa^2 = O(\hat{k}) \), \( \kappa_z^2 = O(\hat{k}) \), \( \alpha = \kappa_z - i\hat{\sigma} + O(\hat{k}) \), and \( \kappa_s = \kappa_m - i\hat{\sigma}_s\hat{h}_s + O(\hat{k}) \), for which \( \hat{\sigma}, \hat{\sigma}_s, \hat{h}_s \) are all of order one. The vertical magnetic field is
\[
\hat{B}_z = -i\hat{F}_z Q/(1 - iQ) + O(\hat{k})
\]
and the electric current is
\[
\hat{J}_y = \hat{E}_y' = -\hat{F}_z/(1 - iQ) + O(\hat{k}).
\]
The induction parameter is
\[
Q = \hat{\sigma}/(1 + \kappa_m - i\hat{\sigma}_s\hat{h}_s) + O(\hat{k}).
\]
Note that (33)–(35) are independent of \( \hat{z} \) and \( \hat{F}_x \). Within the ocean, the horizontal magnetic field varies linearly with depth, \( \hat{z} \). On the surface, \( \hat{z} = 1 \), it is
\[
\hat{B}_x^+ = -\hat{F}_z Q/(1 - iQ) + O(\hat{k});
\]
on the bottom, \( \hat{z} = 0 \), it is
\[
\hat{B}_x^- = \hat{F}_z (\hat{\sigma} - Q)/(1 - iQ) + O(\hat{k}).
\]
The latter is plotted as a function of frequency in Figs. 2 and 3 (long dashes) for the shallow-mantle case. These curves show that the long-wave approximation approaches the solution as \( \omega \to 0 \). The magnetic field in the air, \( \hat{z} > 1 \), for \( \hat{k}(\hat{z} - 1) \ll 1 \), is

\[
\hat{B}_x(\text{air}) = -i\hat{B}_z(\text{air}) = -i\hat{B}_z(\text{ocean}) + o(\hat{k}).
\]

The components of the electric field are:

\[
\hat{E}_x = o(\hat{k}), \quad \hat{E}_y = \hat{B}_z, \quad \hat{E}_z = -\hat{F}_y + o(\hat{k}^2).
\]

The above expressions show that the field induced by long waves is critically dependent on the induction parameter. \( Q \) in turn depends not only on the earth's electrical conductivity structure but also on the time and spatial variation of the water motion, i.e., frequency and wave number, which are related by the propagation equation (6). In the special case, \( \hat{k}\hat{h}_m \gg 1 \), \( \alpha_m \approx 1 \) and the mantle can be ignored. Neglecting sediments, \( Q \approx \frac{1}{2} \hat{\sigma} \). For a deep ocean, then, \( Q \) has the non-negligible value of two. However, as \( h \to 0 \), \( Q \) becomes vanishingly small.

The above expressions also show that the electric current consists of an alternating, almost horizontally uniform, electric-current sheet that is uniform with the ocean depth. There is also induced, in the mantle, an almost horizontally uniform electric-current sheet that is the negative image of the oceanic electric currents. Then the horizontal magnetic field can be obtained, approximately, from \( B_x^+ = \frac{1}{2} \mu I_y \), where \( B_x^+ \) is the magnetic field above a horizontally uniform electric-current sheet, \( I_y \).

The induction parameter, in the superconductive-mantle case without bottom sediments, becomes the real quantity

\[
Q = \hat{\sigma}/(1 + \alpha_a) = \frac{1}{2} \hat{\sigma} [1 - \exp(-2\hat{k}\hat{h}_a)].
\]

For very long waves, \( \hat{k}\hat{h}_a \ll 1 \), the induction parameter by (26), (40) is

\[
Q = \hat{\omega}\hat{\sigma}\hat{h}_a = \hat{\omega}\hat{\sigma}[\hat{h}_m + (\hat{\sigma}_m\hat{\omega})^{-1/2}];
\]

this is an adequate approximation for the shallow-mantle case at low frequency. For example, at 0.1 cph, \( Q = 0.50 \) by (41), \( Q = 0.45 \) by (40), and the amplitude of \( Q \) for a finite conductive mantle is 0.43.

9. Short-crested Waves. The electric current, for a single progressive wave, extends to infinity. For short-crested waves, such is not the case and the following investigates the consequences.

Consider a wave that is a combination of two long-crested progressive waves:
\[ \zeta' = \frac{1}{2} A \exp [i(k_xx + k_yy - \omega t)] , \]
\[ \zeta'' = \frac{1}{2} A \exp [i(k_xx - k_yy - \omega t)] , \]
with wave number \( k^2 = k_x^2 + k_y^2 \). The hydrodynamic equations (1)–(3) are linear. Therefore, the combined wave, \( \zeta = \zeta' + \zeta'' \), is

\[ \zeta = A \cos (k_yy) \exp [i(k_xx - \omega t)] ; \quad (42) \]

it has crests of length \( 2\pi/k_y \). The electromagnetic problem is linear and the solutions are therefore a linear combination of the solutions for \( \zeta' \) and \( \zeta'' \). For example, the horizontal magnetic field for \( \zeta' \) and \( \zeta'' \) is, respectively,

\[ B'_x = \frac{1}{2} b_x(z,k) \exp [i(k_xx + k_yy - \omega t)] , \]
\[ B''_x = \frac{1}{2} b_x(z,k) \exp [i(k_xx - k_yy - \omega t)] , \]

where \( b_x(z,k) \exp [i(kx - \omega t)] \) is the field due to a single progressive wave of amplitude \( A \). The field for the combined wave is then

\[ B_x = \cos (\theta) (B'_x + B''_x) = \cos (\theta) b_x(z,k) \cos (k_yy) \exp [i(k_xx - \omega t)], \]
\[ B_y = \sin (\theta) (B'_x - B''_x) = i \tan (\theta) \tan (k_yy) B_x , \quad (43) \]

where \( \theta = \tan^{-1}(k_y/k_x) \). Likewise, one can show that all of the components for a short-crested wave of amplitude \( A \) are of the same magnitude, to within a factor of \( \cos(\theta) \) or \( \sin(\theta) \), as the components induced by a single long-crested wave of equal amplitude and wave number. The electric current, however, now flows in loops of dimension \( 2\pi/k_x \) by \( 2\pi/k_y \).

10. Tsunami-induced Fields. For purposes of estimating the size of a tsunami-induced electromagnetic field, a modest wave amplitude \( A = 0.2 \) m has been assumed, and the vertical geomagnetic field has been taken to be \( F_z = -0.40 \times 10^{-4} \) weber/m\(^2\)—appropriate for the North Pacific. Tsunami frequencies are low enough so that the horizontal geomagnetic field can be ignored. Table I presents the amplitudes of selected components for a range of tsunami frequencies (1 cph–8 cph). For frequencies of less than 10 cph (Figs. 2 and 3), the other components are related, since then \( |B_z^-| \approx |B_z^+| = |B_x^+|, \)
\( |E_y^-| \approx |E_y^+|, \) and \( |E_y'^-| \approx |E_y'^+| \). The magnetic field in the air near the sea surface is equal to the sea-surface value.

Table I and Figs. 2 and 3 show that the fields vary hardly, with frequency in the tsunami-frequency range. Hence a tsunami of arbitrary wave shape, provided it consists mainly of frequencies 1 cph–10 cph, induces an electromagnetic signal having a similar wave form.

Table I shows that the magnetic field is of the order of 1 gamma (= \( 10^{-9} \) weber/m\(^2\)) and is therefore hopelessly lost in the background ionospheric-
Table I. Amplitudes of tsunami-induced fields ($\gamma = 10^{-9}$ weber/m$^2$).

The amplitude is $A = 0.2$ m and the geomagnetic field is $F_z = 0.4 \times 10^{-4}$ weber/m$^2$. See legend of Fig. 2 for further details.

<table>
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<tr>
<th>Model</th>
<th>Freq (cph)</th>
<th>Electric ($\mu V/m$)</th>
<th>GEK ($\mu V/m$)</th>
<th>Magnetic (gamma)</th>
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</table>

induced noise. For example, magnetic disturbances commonly have variations of 50 gammas in the 1 cph frequency range and occur frequently, especially during times of sun-spot maxima. Thus it appears pointless to attempt detection of tsunami-induced magnetic fields. The electric field, however, appears to be more promising. Deep-sea measurements (e.g., Cox et al., in press) show
magnetic disturbances that give rise to electric fields of the order of \( 1 \mu V/m \). It should be possible, therefore, to detect a tsunami-induced electric field as the predicted values (Table I) are only slightly less than the ionospheric noise. Furthermore, the tsunami-induced field is a narrow-band signal while the ionospheric-induced field is usually a broad-band signal that is coherent over large distances. Therefore it should be possible to remove part of the ionospheric-induced signal from the electric field observed at an oceanic site by use of cross correlation with simultaneously observed electric or magnetic fields at a land site. Electric field recorders on the deep-open sea floor should provide the best observations, but these are difficult to install and maintain. Therefore it seems preferable to first attempt detection of tsunami motion by measuring the GEK field.

Table I also includes the effect of conductive sediments. Their neglect changes the value of the field, at most, by a negligible \( 4\% \). Neglect of self induction and mutual induction is more serious and would lead to an overestimation of the wave amplitude by a factor of two for the GEK in deep water. The long-wave approximation is seen to be completely adequate for describing tsunami-induced fields having frequencies of less than 10 cph.

11. Nonuniform Conductivity. Conductive sediments play a surprisingly minor role. In the present study, electric currents could be produced in the sediments only by mutual induction. These currents turn out to be small for both the deep-ocean and shallow-ocean cases. If the oceanic conductivity is not assumed uniform or if there are vertical nonconducting barriers, such as the continents, the vertical electric current does not vanish; and it is possible, by ohmic means, to pump electric currents into the sediments.

Shallow seas are normally adjacent to continents. Thus the solution derived for the shallow-ocean case is not, in general, applicable for waves that propagate obliquely to the coastline when the offshore distance is small compared with the wavelength. Conductive sediments should also play an important role. This electromagnetic problem is not considered here, since the deep-ocean case is the primary concern of this paper. The present paper shows, however, that the shallow-ocean case with continents can be greatly simplified because self induction and mutual induction can surely be ignored.

The effect of nonuniform conductivity within the ocean itself, on the other hand, is probably minor. For example, the principal conductivity contrast is between the warm water above the thermocline (depth 200 m) and the colder water beneath it. The conductivity contrast is of the order of \( 1 \) ohm\(^{-1}\)m\(^{-1}\). Therefore the conductivity model in the present paper has been modified to include two horizontal uniformly conductive layers within the ocean. The top layer has conductivity \( \sigma_1 \) and thickness \( h \), the bottom layer, \( \sigma_2 \) and \( (1 - h) \). The mean conductivity is \( \bar{\sigma} = \hat{h} \sigma_1 + (1 - \hat{h}) \sigma_2 \). Then, for the top layer,
\[ \chi_1^2 = \hat{k}^2 - i\omega \hat{\sigma}_1 = \chi^2 - i(1 - \hat{h})\omega \Delta \hat{\sigma}, \]  
(44)

and for the bottom layer,

\[ \chi_2^2 = \hat{k}^2 - i\omega \sigma_2 = \chi^2 + i\hat{h}\omega \Delta \hat{\sigma}, \]  
(45)

where \( \Delta \hat{\sigma} = \hat{\sigma}_1 - \hat{\sigma}_2 \) and the mean \( \chi^2 \) is \( h\chi_1^2 + (1 - \hat{h})\chi_2^2 = \hat{k}^2 - i\omega \hat{\sigma} \).

For \( \chi \approx \chi_1 \approx \chi \)—i.e., \( \Delta \chi/\chi \ll 1 \), where \( \Delta \chi = \chi_1 - \chi_2 \)—the vertical conductivity structure can be ignored. Let \( \beta = \omega \Delta \hat{\sigma}/\chi^2 \). Since \( \hat{h} < 1 \), a sufficient condition for \( \Delta \chi/\chi \) to be small is for \( |\beta| \ll 1 \). Then \( \Delta \chi/\chi = -i^2/2 + o(\beta^2) \).

Since \( \chi^2 = k \tanh(k) < \hat{k}^2 \), \( \beta \) is small if

\[ \Delta \hat{\sigma}^2 \ll \hat{\sigma}^2 + \hat{\omega}^2. \]  
(46)

For short waves, \( \hat{\omega} \gg \sigma \), (46) is always satisfied, since \( \Delta \hat{\sigma} < \hat{\sigma} \). For long waves, \( \hat{\omega} \ll \hat{\sigma} \), the last term on the right side of (46) can be ignored. This is a good approximation for frequencies of less than 10 cph. For the typical values, \( \Delta \sigma = 1 \text{ ohm}^{-1} \text{m}^{-1}, \sigma = 3.3 \text{ ohm}^{-1} \text{m}^{-1} \), one then has \( \Delta \sigma/\sigma = 0.3 \) and \( \Delta \chi/\chi \sim 0.15 \). Hence the vertical conductivity structure should play only a minor role. Furthermore, for \( \hat{k} \ll 1 \), the electromagnetic solution for the above case of a two-layer oceanic conductivity structure can be shown to be equal to within \( o(\hat{k}) \) of the solution obtained by ignoring the oceanic conductivity structure, provided, in the latter case, that the mean conductivity is used. That the solutions are comparable is not surprising, because the electric currents are horizontal and parallel to the conductivity structure due to the fact that the emfs that drive them are essentially horizontal and uniform with the oceanic depth. The fields are therefore insensitive to the vertical-conductivity structure.

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