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Tsunami Response at Wake Island: Comparison of the Hydraulic and Numerical Approaches

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ABSTRACT

A comparison is made between two different methods of estimating, at an island, the water-level response associated with incident plane gravity waves of periods characteristic of tsunamis. Van Dorn (1970) has investigated, by means of a laboratory model on an undistorted scale, the amplitude response versus the period and azimuth for Wake Island. In this paper a mathematical model based on a marching numerical integration of the linearized long-wave equations is employed for bathymetry and wave input, duplicating as nearly as possible that of the hydraulic model. Calculations were made for a discrete grid based on an orthogonal "island coordinate" system that facilitates the application of boundary conditions at the island and in the far field. A favorable comparison exists between the results of the two models; the numerical system has a high degree of resolution that verifies details of the response patterns indicated by the hydraulic model.

Introduction. Tsunamis are series of gravity waves of long wavelength compared with depth; they are generated in the ocean overlying strong seismic disturbances. It is certain that no two tsunamis are identical, as indeed no two seismic events are exactly the same. The extent to which common factors exist among tsunamis is largely unknown, since routine observations of the wave trains in a midocean environment are not yet available. Virtually all information about deep-water tsunami characteristics has been inferred from theory and from water-level observations taken at various near-shore recording stations. These investigations indicate that the waves have a broad-band spectrum, with periods ranging from several minutes to hours and with wave heights of the order of one meter in oceanic waters. It is precisely the uncertainties in individual wave properties that has produced considerable interest in hydrographs recorded at stations located on Pacific Ocean islands.

Basic information about the spectral distribution of tsunami energy has been sought from recordings made at regular tide stations on continental and island shores. Within the last two decades, the correlation of large numbers of these...
results has indicated that the hydrographs primarily contain the characteristics of the localized resonant response to a tsunami. In view of this evidence, the recordings made at an island that has limited spatial dimensions and is well separated from any neighboring islands should minimize the masking of the incident wave train by island-wave interactions. Wake Island, shown in Fig. 1, is an isolated coral atoll of this nature. Four recent recordings by Van Dorn (1965) at Wake Island have presented the first opportunities to examine a tsunami in such a nearly oceanic setting.

In the case of tsunami modification induced by an island, it is anticipated that the effects of scattering and the excitation of possible trapped modes (Longuet-Higgins 1967) are governing factors. The numerical study described in this paper has examined the local effects of scattering by an island in terms of the response patterns generated by plane simple harmonic wave trains that have periods representative of tsunamis. An hydraulic study by Van Dorn (1970) has investigated the magnitude of the induced response at Wake Island. The numerical work was performed by us in order to form a basis for a comparison with the hydraulic study. The purpose of the comparison is to establish a coherence of results which, within the limitations of the models, demonstrates a consistent representation of the interaction of long waves with islands. This verification of the numerical system is a prelude to further investigations of the relationship between near-shore tsunami recordings and the oceanic character of the wave trains and to further studies relating response patterns and observed run-up heights that have been gathered for a number of Pacific Ocean islands.
The Numerical Model. The mathematical basis for the numerical approach is the classical linearized longwave equation,

$$\frac{\partial^2 \zeta}{\partial t^2} = g \nabla \cdot (D \nabla \zeta), \quad (1)$$

together with appropriate boundary conditions on the island shore and in the far field. In eq. (1), \( \zeta \) is the free-surface elevation above the equilibrium level, \( t \) is time, \( D \) is the local still-water depth, \( g \) is the acceleration due to gravity, and \( \nabla \) denotes the horizontal-gradient operator.

For the island shore, we require

$$\mathbf{n} \cdot \nabla \zeta |_{D=0} = 0, \quad (2)$$

where \( \mathbf{n} \) is a horizontal unit vector normal to the shore boundary. If a vertical barrier exists with \( D \neq 0 \), then (2) would demand that the normal derivative of \( \zeta \) vanish. On the other hand, for a beach, the mean shoreline is defined by \( D = 0 \); (2) would then demand only that the normal derivative of \( \zeta \) be finite. This rules out solutions for which \( \zeta \) has a logarithmic singularity at the shore. Condition (2), in either case, implies zero net flux of energy at the shoreline (i.e., total reflection).

In the far field, on the other hand, we require that the scattered part of the wave energy be outwardly propagating. Letting \( \zeta_i \) denote the incident wave field, the far-field radiation condition can be expressed in the form

$$r^{1/2} (\zeta - \zeta_i) \approx \text{constant} \quad (3a)$$

along the characteristic path,

$$\frac{dr}{dt} = (gD_\infty)^{1/2}, \quad \frac{d\theta}{dt} = 0 \quad (3b)$$

for large \( r \). Here \((r, \theta)\) are polar coordinates, with origin in the island, and \( D_\infty \) is the far-field depth, assumed constant. The incident field of \( \zeta \) is prescribed as a simple-harmonic plane progressive wave of unit amplitude. It is a solution of (1) for \( D = D_\infty \). In polar coordinate form, an incident field satisfying these conditions is

$$\zeta_i = \sin \Phi(r,t), \quad (4a)$$

where

$$\Phi = \omega [t + (r - R)(gD_\infty)^{-1/2} \cos (\theta - \theta_0)]. \quad (4b)$$

In (4b), \( \theta_0 \) is the azimuth from which the incident waves arrive, \( \omega \) is the (radian) frequency, and \( R \) is a constant reference radius.

We seek solutions for (1), (2), (3) with input (4), which are periodic in time, with frequency \( \omega \). Physically, such solutions correspond to the forced steady
oscillations attained after any initial transient disturbance has radiated well away from the island. Accordingly, the initial conditions on $\zeta$ are arbitrary. In the numerical solutions discussed in this work, we took $\zeta$ and $\partial^2 \zeta / \partial t^2$, initially zero, for all $\theta$ and $r \leq R$ (but external to the island). Moreover, $\zeta_t$ was evaluated by (4a,b), provided $\Phi > 0$; otherwise $\zeta_t = 0$. This implies that the incident wave has a "front" that initially lies outside the radius $R$, for which solutions are sought.

For the special case of an axially symmetric island, with $D$ a function of $r$ only, it is natural to pose (1) and (2) in polar coordinates. Two examples for which analytical solutions are available were employed by Vastano and Reid (1967) to verify a marching numerical-integration procedure in a polar grid.

Although the study of the response at Wake Island could have been carried out in a polar coordinate system, further precision has been introduced into the numerical model by the adoption of a special orthogonal-coordinate system (Reid and Vastano 1966). This system, hereafter referred to as island coordinates, can be considered as a perturbed polar system $(\rho, \beta)$, in which the isolines of $\rho$ are closed curves, one of which, $\rho = 1$, represents an island shoreline of arbitrary shape. In the far field, the isolines of $\rho$ approach a circular form. The evaluation of an appropriate conformal mapping relation having these properties for a given island is summarized in Appendix A. In Fig. 2, an example of island coordinates with constant $(\rho, \beta)$ spacing is shown for Wake Island.

In terms of island coordinates, (1) takes the form

$$\frac{\partial^2 \zeta}{\partial t^2} = g s^2 \left[ \rho \frac{\partial}{\partial \rho} \left( \rho D \frac{\partial \zeta}{\partial \rho} \right) + \frac{\partial}{\partial \beta} \left( D \frac{\partial \zeta}{\partial \beta} \right) \right],$$

in which $s$ is a $(\rho, \beta)$-dependent scale factor defined by

$$s = |\nabla \ln \rho| = |\nabla \beta|.$$

Specific relations for the evaluation of the function $s$ are given in Appendix A. The transformation employed is such that $s \rightarrow r^{-1}$ while $r/\rho$ and $\theta - \beta$ approach constant values for large $\rho$. Thus (5) takes on a polar coordinate form for large $\rho$.

The finite-difference approximation of the wave equations could be formulated in a grid system of constant increments of $(\rho, \beta)$. However, there are distinct advantages in adopting variable increments of $\rho$. Systematic selection of the $\rho$ spacing was made in accord with the travel time, $\mu$, for long-wave propagation over the average depth profile starting at the island shoreline. By adopting a grid based on constant $\Delta \mu$ spacing, the resolution is enhanced in the shallow-water region near the island, where the greatest amplitude modification occurs. The dashed curves shown in Fig. 2 are travel-time grid lines based on constant $\Delta \mu$ spacing.
The coordinate transformation relating $\varrho$ and $\mu$ is defined by

$$\mu = \int_{1}^{\varrho} \frac{h(\varrho)}{\varrho} \, d\varrho,$$

where

$$h(\varrho) = \left[ \frac{s}{(gD)^{1/2}} \right]^{-1}.$$  \hspace{1cm} (8)

The overbar in (8) indicates an average of a quantity over the full range of $\beta$ at a fixed $\varrho$ value. For the special case of a circular island of radius $r_0$, the above relations reduce to

$$\mu = \int_{r_0}^{r} (gD)^{-1/2} \, dr.$$  \hspace{1cm} (9)

The wave equation, expressed in terms of the independent coordinates $(\mu, \beta, t)$, has the form

$$\frac{\partial^2 \zeta}{\partial \tau^2} = g s^2 \left[ h \frac{\partial}{\partial \mu} \left( hD \frac{\partial \zeta}{\partial \mu} \right) + \frac{\partial}{\partial \beta} \left( D \frac{\partial \zeta}{\partial \beta} \right) \right].$$  \hspace{1cm} (9)

The boundary condition (2) in this system is simply

$$\frac{\partial \zeta}{\partial \mu} = 0, \quad \text{at} \quad \mu = 0.$$  \hspace{1cm} (10)
It is noteworthy that (10) holds for a beach condition \((D = 0)\) as well as a vertical barrier (see Appendix B for proof). Thus the use of the travel-time coordinate, \(\mu\), overcomes the restriction as to the type of shore boundary that has been indicated by Lautenbacher (1970). Any difference in the behavior of the oscillations of the free surface near the shore, for the barrier or beach, is governed by the wave equation (9) with the appropriate depth field.

The far-field condition in the \((\mu, \beta, t)\) coordinate is

\[
 r^{1/2} (\zeta - \zeta_t) \approx \text{constant} \tag{11a}
\]

along the path

\[
 \frac{d\mu}{dt} = 1, \quad \frac{d\beta}{dt} = 0 \tag{11b}
\]

for large \(r\) (hence large \(\mu\)).

In the numerical-integration procedure, we employed a centered finite-difference analog of (9) at uniform time steps for all interior points on a grid having uniform increments of \(\mu\) and \(\beta\). A one-sided second-order finite-difference version (10) was employed in evaluating the shoreline values of \(\zeta\) at each time step (see Appendix B). The far-field condition (11) is necessarily applied at a finite but large value of \(\mu\) (corresponding approximately to \(r = R\)). The application of (11) involves a spatial interpolation at time \(t\) to evaluate \(\zeta\) at the outer \(\mu\) contour at \(t + \Delta t\) (see Vastano and Reid 1967 for details of this procedure). Finally, appropriate equivalence requirements are imposed on \(\zeta\) with respect to its periodic dependence on the cyclic coordinate, \(\beta\).

The grid network was chosen on the basis of a mandatory compromise between (i) the angular resolution of the incident wave system, (ii) the far-field requirement for the outer boundary condition, and (iii) the available computer storage. The grid had 54 radial steps and 90 angular steps in the \((\mu, \beta)\) coordinates. The 54 radial increments correspond to a constant \(\mu\) increment of 14.5 seconds; their total provides a radial extent of approximately 132 km from the reefline to the outermost grid line. The depth field was taken as that of the hydraulic Wake Island model investigated by Van Dorn (1970).

The prototype time increment consistent with computational stability was found by calculating the travel time for a long wave passing diagonally through each \((\mu, \beta)\) grid section (see Vastano and Reid 1967). A time increment of one second was selected by truncating the minimum travel time. This limit was encountered at the inner boundary of the grid system in the vicinity of Peacock Point on the southeasternmost side of Wake Island (Fig. 1). The value is indicative of the high resolution afforded by island coordinates.

The computations were started with two successive initial fields of \(\zeta\) set equal to zero and equal to the incident wave front that was just entering the outer boundary of the numerical grid. The azimuth of the incident waves was taken as 72 degrees counterclockwise from the east—in conformity with the conditions of the Van Dorn (1970) model. In the discussion that follows
(Comparison of Models), all azimuths are referenced in this manner. Computations were continued until the oscillations of $\zeta$ at the shoreline were judged to have reached a stable amplitude under the influence of the sustained monochromatic plane-wave input. Most runs required about 70 minutes of machine time on an IBM 7094 computer. The amplitudes and phase-lags at the island shore were evaluated from the output time sequences for each of the shore grid points.

Comparison of Models. The response at Wake Island has been analyzed for a number of periods in the tsunami range. Figs. 3 and 4 present the relative amplitude and phase-lag relationships of the response patterns for three of the periods common to the numerical and hydraulic models: 362, 542, and 835 seconds. A striking feature in the comparison of these amplitudes is the accord in the relative positions of the maxima and minima of the hydraulic and numerical patterns. The high resolution of the numerical system is demonstrated by the ability of the model to predict the small sheltering afforded by Peacock Point, which lies at an azimuth of 314°. Both models consistently predict this feature of the response.

The phase-lag graphs for the numerical model have been drawn relative to the far-field wave at an azimuth of 162°. This azimuth is 90° from the approach azimuth of the incident wave; in all three graphs it is the angle at which a significant increase in the phase lag is evident. Unfortunately, the phase measurements from the hydraulic model are not available for comparison. For the longest period, 835 seconds, the phase of the numerical-response patterns at the shoreline leads the far-field wave for azimuths from approximately 0° to 148°. For the other two periods there is a shift of the curves toward the phase lag. The effect of Peacock Point at the 314° azimuth is strongly evident in respect to the abrupt decrease in the phase lag as the Point is rounded from west to east.

The consistent trend of the numerically evaluated amplitudes to exceed the hydraulic counterparts may be attributed to the existing differences in the two models. The numerical model is expressed in the island-coordinate system, which is essentially polar in the far field. Full representation of the region covered by the hydraulic model in this system would require a radial coverage that exceeds the available computer storage. As a result, the outer-boundary condition of the numerical model is taken as radiational while the hydraulic model has a far-field shoreline. The second dissimilarity concerns the forcing function. The numerical system utilized a polar representation of a plane monochromatic wave traveling past the grid system. The specification of the plane wave at the outer boundary constitutes a numerical forcing function that is assigned an amplitude of unity. In the hydraulic model, the incident wave trains were produced by the vertical motion of a finite-length generator bar. Differences in the observed and calculated response patterns may be influenced
Figure 3. Relative amplitude vs. azimuth for three response patterns at Wake Island. The solid lines represent the numerical solution, the open circles the hydraulic-model results. The azimuth of incident waves is from 72°.

by diffractional spreading of the incident waves from the generator bar. This effect is not accounted for in the forcing function used in the numerical model.
An additional possible source of difference is that the island measurements in the hydraulic model were actually offshore from the outer reef while the numerically deduced island response corresponds to the position of the outer reef (a vertical barrier in both models). In terms of the \( \mu \) coordinate, the difference in position is only about one \( \mu \) increment; however, it may explain partially why the numerically deduced amplitude response exceeds that from the hydraulic model.

We consider the comparison of the two approaches to be a satisfactory one. The response patterns exhibit a similarity that indicates a consistent representation of tsunami-island interaction, at least within the limitations discussed above. Further applications of the numerical system are currently planned; these will provide an opportunity to relate the numerically derived response patterns to field data.
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APPENDIX A

The Island-coordinate System.

A conformal mapping relation, \( z = f(w) \), is required such that a unit circle in the \( w \) plane is mapped as a closed curve, \( \Gamma \), corresponding to the prescribed island shoreline in the \( z \) plane. Moreover, each point exterior to the unit circle of the \( w \) plane is to map, one to one, into a corresponding point exterior to the island boundary in the \( z \) plane. Let \( r, \rho \) be the moduli of the complex numbers \( z, w \) and let \( \theta, \beta \) be their arguments. Thus

\[
z = r e^{i\theta} \quad \text{and} \quad w = \rho e^{i\beta}, \tag{A-1}
\]

the origin being centered within the island in both planes.

The transformation \( f(w) \) should permit the island perimeter to correspond to \( \rho = 1 \) and, as \( \rho \to \infty \), \( \ln \rho \) should approach a linear function of \( \ln w \). This implies that, in the far field, the coordinates \((\rho, \beta)\) correspond to polar coordinates with \( r \) proportional to \( \rho \) and with \( \theta - \beta \) a constant. For convenience in the analysis, \( r \) is expressed in normalized units such that the perimeter \( L \) has
the value of $2\pi$, which corresponds to the principal range of $\theta$ or $\beta$. A transformation that possesses the desired characteristics is

$$\ln z = \ln w + \sum_{n=0}^{N} C_n w^{-n},$$  
(A-2)

where the $C_n$ are complex constants, and $N$ is a suitably large number (Lamb 1945: 75). This form is applicable to the exterior mapping problem in which $|w^{-n}| \to 0$ as $\varrho \to \infty$ for all $n > 0$. Letting $C_n = A_n + iB_n$, where $A_n, B_n$ are real, then, at the shoreline ($\varrho = 1$), eq. (A-2) can be separated into the component relations

$$\ln r = A_0 + \sum_{n=1}^{N} (A_n \cos n\beta + B_n \sin n\beta),$$

$$\theta - \beta = B_0 + \sum_{n=1}^{N} (B_n \cos n\beta - A_n \sin n\beta).$$  
(A-3)

Consider that the $r$ and $\theta$, corresponding to the island shoreline, are specified parametrically in terms of a normalized arc length, $l$, measured along $I'$ from some fixed point on $I$.

The problem is to find that set of coefficients, $A_n, B_n$ and an unknown function $\beta(l)$ or $l(\beta)$, such that relations (A-3) are satisfied in a least-squares sense, for given $r(l)$ and $\theta(l)$. This requires an iterative procedure starting with the approximation $\beta = l$ and employing both relations in (A-3) to obtain successive approximation of $A_n, B_n$ and $\beta(l)$ or $l(\beta)$. The details of a stable iterative procedure have been given by Reid and Vastano (1966). In this procedure we choose the number of terms, $N$, such that the mean-square error of fit of the island boundary falls below some preselected level. For the Wake Island fit, $N$ is of the order of 50. The coefficient $B_0$ was taken such that $\beta = \theta = 0$ at the boundary. An alternative is to take $B_0 = 0$, in which case $\beta = \theta$ for large $\varrho$.

Once the Fourier coefficients are known, the coordinate systems $(\varrho, \theta)$ and ($r, \theta$) are related by

$$\ln r = \ln \varrho + A_0 + \sum_{n=1}^{N} (A_n \cos n\beta + B_n \sin n\beta) \varrho^{-n},$$

$$\theta = \beta + B_0 + \sum_{n=1}^{N} (B_n \cos n\beta - A_n \sin n\beta) \varrho^{-n}.$$  
(A-4)

The $(\varrho, \beta)$-dependent scale factor, $s$, can be evaluated directly from the complex mapping relation (A-2) as $|d\ln w/dz|$, taking into account the dimensions of $z$. Using (A-1) and (A-2), it follows that

$$s = \frac{1}{r} \left| 1 - \sum_{n=1}^{N} nC_n \varrho^{-n} e^{in\beta} \right|^{-1},$$  
(A-5)
where $r$ is the dimensional radial distance related to $(\varrho, \beta)$ by

$$ r = \frac{L}{2\pi} \varrho \exp \sum_{n=0}^{N} (A_n \cos n\beta + B_n \sin n\beta) \varrho^{-n}, \quad (A-6) $$

where $L$ is the dimensional perimiter of the island boundary.

**APPENDIX B**

*Coastal Boundary Condition.*

In terms of the $(\varrho, \beta)$ coordinates at $\varrho = 1$,

$$ \mathbf{n} \cdot \nabla \zeta = \varrho \frac{\partial \zeta}{\partial \varrho}. \quad (B-1) $$

However, in the $(\mu, \beta)$ coordinates, it follows from (7), (8) and the above relations that

$$ \frac{\partial \zeta}{\partial \mu} = \frac{s}{\bar{s}} \bar{s} \frac{(g \bar{D})^{1/2}}{s} \mathbf{n} \cdot \nabla \zeta \quad (B-2) $$

on $\varrho = 1 \ (\mu = 0)$. We assume that $\bar{s}/s$ is not singular and that $\mathbf{n} \cdot \nabla \bar{D}$ is finite and not zero.

For a vertical barrier, $\mathbf{n} \cdot \nabla \zeta$ must vanish (implying zero normal flow) and hence (B-2) leads to

$$ \frac{\partial \zeta}{\partial \mu} = 0. \quad (B-3) $$

For a beach, $D = \bar{D} = 0$, and, if $\mathbf{n} \cdot \nabla \zeta$ is finite, then (B-2) again reduces to (B-3).

In the present study a one-sided finite difference version of (B-3) was employed. An approximation that has an accuracy comparable to that of the finite-difference version of (9) is

$$ \zeta(0, \beta, t) = \frac{4}{3} \zeta(\Delta \mu, \beta, t) - \frac{1}{3} \zeta(2 \Delta \mu, \beta, t). \quad (B-4) $$

This follows from (B-3), using Newton's interpolation formula for $\zeta$, accurate to second degree in $\mu$. 