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Eddy Friction in the Ocean

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ABSTRACT

The scalar energy spectrum of steady isotropic two-dimensional turbulence is derived for two-dimensional motion in a homogeneous ocean in which the dissipation occurs in a bottom-friction layer. The significant property of the spectrum is a range of wave numbers in which the mean-squared vorticity is conserved and energy is transferred to the friction layer. One feature of the horizontal turbulence is a characteristic period that is much greater than the response time of the friction layer. Specific results have been obtained on the representation of turbulence in numerical ocean models.

1. Introduction. In this paper a set of relationships is derived for the two-dimensional motion in a homogeneous ocean. These relationships make it possible to predict the properties of the fluid on a scale that is less than the scale of resolution of the mean motion. Such properties characterize the “two-dimensional turbulence” of the fluid. The essential point of the analysis is that spectral energy, by means of a bottom-friction layer, is extracted from all wave numbers of the horizontal motion in the interior of the fluid. If it is assumed that the 2-D turbulence is steady, homogeneous, and isotropic and that its properties are determined by the local net-energy transfer rate, then an equation for the scalar energy spectrum of the 2-D turbulence is obtained. Specific deductions have been made to obtain the coefficient of effective lateral-eddy viscosity (§ 4) as well as the representation of turbulence in numerical ocean models (§ 5). Finally, the application of the analysis under less stringent assumptions is discussed (§ 6).

2. Dissipation in the Bottom-friction Layer. If we suppose that the fluid is homogeneous, with negligibly thin surface-friction and bottom-friction layers, then the bottom stress, $T_B$, has the form

$$T_B = q_0 f(Q^2) Q,$$  

(2.1)

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where

\[ \mathcal{Q}^2 = (U + u)^2 + (V + v)^2 ; \]

here \( U, V \) are the \( x \) and \( y \) components of the mean velocity averaged over depth, \( u, v \) are the \( x \) and \( y \) components of the instantaneous fluctuations in velocity due to horizontal turbulence, \( \rho_o \) is the density, and the function \( f(\mathcal{Q}^2) \) defines the bottom-friction law.

Similarly, the \( x \) and \( y \) components of bottom stress are

\[
T_{Bx} = \rho_o f(\mathcal{Q}^2)(U + u), \tag{2.2}
\]

\[
T_{By} = \rho_o f(\mathcal{Q}^2)(V + v). \tag{2.3}
\]

Hence, we obtain, for the components of the total frictional stress in the integrated momentum equations,

\[
\frac{F_x}{\rho_o} = f(\mathcal{Q}^2)(U + u) + H \left( \frac{\partial}{\partial x} \bar{u}^2 + \frac{\partial}{\partial y} \bar{u}v \right), \tag{2.4}
\]

\[
\frac{F_y}{\rho_o} = f(\mathcal{Q}^2)(V + v) + H \left( \frac{\partial}{\partial x} \bar{u}v + \frac{\partial}{\partial y} \bar{v}^2 \right); \tag{2.5}
\]

where \( H \) is the depth of the fluid, the overbar denotes an average over a fundamental interval, and \( \bar{u}^2, \bar{v}^2, \) and \( \bar{uv} \) are the horizontal Reynolds stresses.

Consider next the energy dissipation in the bottom-friction layer. Suppose that this layer is statistically homogeneous and that the dissipation occurs sufficiently close to the bottom so that the stress is constant. Then the mean rate of dissipation, \( \varepsilon \), in the layer, averaged over the column of fluid, is

\[
\varepsilon = \frac{1}{H} \frac{i}{f(\mathcal{Q}^2) \mathcal{Q}^2}; \tag{2.6}
\]

(2.6) is applicable, provided that the characteristic period of the horizontal turbulence is much greater than the response time of the friction layer. This assumption is justified, \textit{a posteriori}, in § 3.

By specializing the bottom-friction law with the power-law form

\[
f(\mathcal{Q}^2) = K(\mathcal{Q}^2)^n, \tag{2.7}
\]

where \( n \) is a non-negative constant and \( K \) is a constant with dimensions \([LT^{-1}]^{2-2n}\), \( \varepsilon \) may be evaluated as follows. Substituting (2.7) in (2.6), we obtain

\[
\varepsilon = \frac{K}{H} \frac{i}{((U + u)^2 + (V + v)^2)^{n + 1}}; \]

by expanding to terms of the first order in the horizontal turbulence,
\[ \varepsilon = \frac{K|Q|^{2n+2}}{H} \left[ 1 + (n+1) \left( 1 + 2n \frac{U^2}{Q^2} \right) \frac{\overline{u^2}}{Q^2} + (n+1) \left( 1 + 2n \frac{V^2}{Q^2} \right) \frac{\overline{v^2}}{Q^2} + 4n(n+1) \frac{UV \overline{uv}}{Q^4} \right] \]  

(2.8)

where \( Q = (U^2 + V^2)^{1/2} \). Eq. (2.8) may be considerably simplified for isotropic turbulence, in which

\[ \overline{u^2} = \overline{v^2} = \frac{1}{2} \overline{u^2}, \quad \overline{uv} = 0; \]

this yields

\[ \varepsilon = \frac{K|Q|^{2n+2}}{H} \left[ 1 + m \xi^2 \right], \quad \xi \to 0, \]

(2.9)

where \( m = (n+1)^2 \), and \( \xi = (\overline{u^2})^{1/2}/|Q| \) is the turbulent intensity. Hence, for steady homogeneous isotropic turbulence, the rate of dissipation consists of two components—the dissipation due to the mean motion, \( \varepsilon_M \), and the dissipation due to the turbulent motion, \( \varepsilon_T \), where,

\[ \varepsilon_M = g_n Q^2, \]

(2.10)

\[ \varepsilon_T = mg_n \overline{u^2}, \]

(2.11)

\[ g_n = \frac{K|Q|^{2n}}{H}. \]

(2.12)

Examine the relative rates of dissipation. The essential effect of increasing the exponent, \( n \), in the bottom-friction law is to increase the component of dissipation due to turbulent motion. This is seen clearly in two special cases of physical importance, namely linear bottom friction and quadratic bottom friction. For linear bottom friction,

\[ n = 0, \quad m = 1, \quad g_0 = \frac{K}{H}, \]

(2.13)

and for quadratic bottom friction

\[ n = \frac{1}{2}, \quad m = \frac{9}{4}, \quad g_{1/2} = \frac{K|Q|}{H}. \]

(2.14)

The former corresponds to a constant vertical-eddy viscosity in the friction layer and the latter represents observations in tidal channels, where approximately

\[ K = 0.0025. \]

(2.15)
These observations have been discussed by Proudman (1953: 136, 310, 316). Note, however, that $K$ depends on (i) the exact distance from the bottom at which $Q$ is measured and (ii) on the roughness of the bottom.

The turbulent stresses also may be expressed in a similar manner. Substituting (2.7) in (2.2) and (2.3), we obtain

$$T_{Bx} = \varrho_o KQ |Q|^2 n U \left[ 1 + n \left( 3 + 2(n-1) \frac{U^2}{Q^2} \right) \frac{u^2}{Q^2} + n \left( 1 + 2(n-1) \frac{V^2}{Q^2} \right) \frac{v^2}{Q^2} \right] +$$

$$+ \varrho_o K |Q|^2 n V \left[ 2n \left( 1 + 2(n-1) \frac{U^2}{Q^2} \right) \frac{uv}{Q^2} \right] + \ldots$$

and an analogous expression for $T_{By}$.

Here, for isotropic turbulence, we obtain

$$\frac{F_x}{\varrho_o} = KQ |Q|^2 n U [1 + l \xi^2], \quad \xi \to 0 \quad (2.16)$$

$$\frac{F_y}{\varrho_o} = KQ |Q|^2 n V [1 + l \xi^2], \quad \xi \to 0 \quad (2.17)$$

where $l = n(n + 1)$; the Reynolds stress terms have vanished identically. Eqs. (2.16), (2.17), together with (2.9), completely describe 2-D isotropic horizontal turbulence, provided that $\xi$ may be estimated from the properties of the mean flow. The next section introduces a dimensional hypothesis that makes this possible.

3. The Two-dimensional Scalar Energy Spectrum. Consider now the turbulent energy equation for the 2-D motion. Assuming that the rate of dissipation outside the bottom-friction layer is zero, we obtain for steady homogeneous turbulence,

$$\varepsilon = -\bar{u} \frac{\partial U}{\partial x} - \bar{v} \frac{\partial V}{\partial y} - \bar{uv} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right), \quad (3.1)$$

where $\varepsilon$ is the net energy-transfer rate to the bottom-friction layer and the right-hand side is the rate of working by the Reynolds stresses. In the isotropic range considered in § 2, the rate of working is zero and

$$\varepsilon = \varepsilon_T = mg_u \bar{u}^2. \quad (3.2)$$

Eq. (3.2) is the fundamental relation of the analysis. By defining the scalar energy spectrum with the relation

$$\frac{1}{2} \bar{u}^2 = \int_0^\infty E(k) \, dk, \quad (3.3)$$
where $k$ is a scalar wave number, by substituting (3.3) in (3.2), and by differentiating, we obtain the analogous spectral-energy density equation

$$-rac{\partial}{\partial k} \varepsilon(k) - 2mg_n E(k) = 0, \quad k \gg k_0;$$

(3.4)

(3.4) is applicable in the range $k \gg k_0$, where $k_0$ is a wave number characteristic of the energy supply derived from the mean motion. Here $\varepsilon(k)$ represents the net rate of spectral-energy transfer from all wave numbers whose magnitude is smaller than $k$ to those greater than $k$; the negative sign indicates that the second term represents the dissipation of energy in the friction layer. This range, $k \gg k_0$, which is characterized by an extraction of spectral energy from the 2-D motion to the 3-D turbulence that effects the dissipation, will be called the energy-extraction range. Consider now the form of $E(k)$ in this range. Assume that the spectral energy-transfer process is local in wave-number space; this leads to the hypothesis that, for a homogeneous fluid, $E(k)$ must be a function of $\varepsilon(k)$ and $k$ only. Hence, on dimensional reasoning, $E(k)$ has the form

$$E(k) = A \varepsilon(k)^{2/3} k^{-5/3}, \quad k \gg k_0,$$

(3.5)

where $A$ is a constant, which hopefully is of order unity. Eq. (3.5) is an expression analogous to the Kolmogoroff spectrum for the inertial subrange in three-dimensional turbulence; we may expect this expression to be applicable only if the length scale associated with the bulk of the energy containing eddies of the 2-D motion is well separated from the length scale of the energy-extraction range, i.e., $k \gg k_0$. Substituting (3.5) into (3.4), we obtain the following relations for the energy-extraction range.

First, by integrating (3.4) with respect to $k$ and by satisfying the integral condition (3.2), we obtain the net energy-transfer rate

$$\varepsilon(k) = m^3 A^3 g_n^3 k^{-2}$$

(3.6)

and the scalar energy density

$$E(k) = m^2 A^3 g_n^2 k^{-3}.$$  

(3.7)

Second, the turbulent kinetic energy is

$$\frac{1}{2} \overline{u^2} = \frac{1}{2} m^2 A^3 g_n^2 k^{-2}$$

(3.8)

and the turbulent intensity is

$$\xi = m A^{3/2} g_n k^{-1} |Q|^{-1}.$$  

(3.9)
Finally, the net rate of transfer of the mean-squared verticity, defined by the relation \( \eta = k^2 \mathbb{E}(k) \), and the characteristic period of the 2-D turbulence, defined by the relation \( \chi = k^{-1} (\mathbb{u}^2)^{-1/2} \), are both constants, where

\[
\eta = m^3 A^3 g_n^3 \tag{3.10}
\]

and

\[
\chi = m^{-1} A^{-3/2} g_n^{-1} = A^{-1/2} \eta^{-1/3} \tag{3.11}
\]

Using (3.11), we may now examine the assumption in §2 that the characteristic period of the horizontal turbulence is much greater than the response time of the friction layer. Consider a quadratic bottom-friction layer, which arises from similarity arguments and is a good approximation to observation. The response time of this layer is of the order \( U_x^{-1} Z \), in which \( U_x = |T_B/G_0|^{1/2} \) is the friction velocity and \( Z \) is the height from the bottom, such that

\[
0 < U_x^{-1} Z < |f^{-1}|; \tag{3.12}
\]

here \( f \) is the Coriolis parameter. Hence, substituting in \( U_x \) from (2.1), the ratio of the response time to the characteristic period of the horizontal turbulence is given by

\[
\chi^{-1} U_x^{-1} Z = m A^{3/2} K^{1/2} \left( \frac{Z}{H} \right) < m A^{3/2} K \left| \frac{Q}{Hf} \right|. \tag{3.13}
\]

Substitution of typical values for the parameters into (3.13) shows that, in all basins of oceanographic importance,

\[
\chi^{-1} U_x^{-1} Z \ll 1. \tag{3.14}
\]

Indeed, it appears from (3.13) that the thickness of the friction layer need not even be much less than the depth to make (3.14) applicable. It is concluded, therefore, that the assumption in §2 is justified rather widely in oceanographic and limnological situations. Also, it may be valid in basins where the thickness of the dissipation layer is an appreciable fraction of the total depth (provided that there is due modification for the variation in horizontal velocity with height). However, this situation will not be considered further.

The above relations completely describe the properties of the 2-D horizontal turbulence in the energy-extraction range.

4. Lateral-eddy Viscosity in an Isotropic Fluid. In principle, we may expect the isotropic dissipation process described above to give rise to a coefficient of effective eddy viscosity associated with the mean horizontal motion. This coefficient, which is not related in any way to the horizontal shears, reflects the mutual interaction of the smaller eddies and the larger eddies in the 2-D turbulence; this is analogous to the coefficient of effective eddy viscosity
associated with the energy cascade in a three-dimensional isotropic turbulence (Batchelor 1956: 129–130). The form of the coefficient can be deduced as follows. Consider the component of the mean motion, with wave number $k'$:

$$U(k') = U_0 e^{i(ax + by)}, \quad k > k' > k_0$$ (4.1)

$$V(k') = V_0 e^{i(ax + by)},$$ (4.2)

where $k' = (a^2 + b^2)^{1/2}$. The frictional stresses due to this component, by using (2.16), (2.17), are

$$F_x = gn U(k') \{1 + l \xi^2\},$$ (4.3)

$$F_y = gn V(k') \{1 + l \xi^2\}.$$ (4.4)

Invoking (3.10), and substituting (4.1) and (4.2) into (4.3) and (4.4), we obtain

$$\frac{F_x}{\rho_o H} = -N(k') \nabla^2 U(k')$$ (4.5)

$$\frac{F_y}{\rho_o H} = -N(k') \nabla^2 V(k')$$ (4.6)

where

$$N(k') = m^{-1} A^{-1} \eta^{1/3} k^{-2}(1 + l \xi^2)$$ (4.7)

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. $$

Thus the frictional stresses due to a wave number, $k'$, of the mean motion are formally equivalent to the lateral stresses on the right-hand side of (4.5), (4.6), in which the effective coefficient of lateral-eddy viscosity, $N(k')$, is defined by (4.7).

For $k \to \infty$, (4.7) reduces to the relation

$$N(k') = B \eta^{1/3} k'^{-2},$$ (4.8)

where $B = m^{-1} A^{-1}$, which applies to each wave number of the motion and is the form given by Crowley (1968) on dimensional grounds.

5. Physical Interpretation of the Energy Spectrum. The results of the above analysis are consistent with the quasisteady state for 2-D turbulence postulated by Kraichnan (1967); thus an approximate $k^{-3}$ vorticity-transfer range carries the mean-squared vorticity input up to $k > k_0$. The approximate $k^{-5/3}$ energy-transfer range, also postulated by Kraichnan, would carry most of the input
down to zero wave number, but this is not essential because it is shown that energy may be extracted from the 2-D motion by a friction layer at all wave numbers $k \gg k_0$ and that the energy would subsequently be dissipated. The 2-D turbulence in the interior does indeed have the characteristics of a cascade of means-squared vorticity, $\eta$, with a net energy-transfer rate, $(3.6)$,

$$\varepsilon (k) \propto k^{-2} \quad k \gg k_0.$$ 

The exact form of the bottom-friction law is not critical. However, the case of quadratic bottom friction is especially pertinent. Here (3.9) and (3.11) yield

$$\zeta = m A \frac{3}{2} K H^{-1} k^{-1}, \quad (5.1)$$

$$\chi = m^{-1} A^{-3/2} K^{-1} H |Q|^{-1}. \quad (5.2)$$

Hence the turbulent intensity is proportional to the reciprocal of the wave number, $k^{-1}$, and the characteristic period of the turbulence is proportional to $|Q|^{-1}$. This suggests vortex-like motion, the strength of which is proportional to the mean velocity and the size of which is limited by the restriction that $k \gg k_0$. Consider now the implications of a quadratic friction layer in ocean-circulation models.

6. Application to Ocean-circulation Models. A practical application of the analysis is in numerical models of oceanic circulation. Consider first the steady-state models. Suppose that the scale of resolution, $k_g$, of the model is given by the relation

$$k_g = \frac{\pi}{h}, \quad k_g \gg k_0, \quad (6.1)$$

where $h$ is a finite-difference grid length; i.e., eddies of a half wavelength less than the grid interval are not represented explicitly in the solution. And assume that the turbulent intensity, $\zeta_g$, not explicitly represented by the solution, is sufficiently small for application of the analysis. Then, by (5.1) and (6.1),

$$\zeta_g = \frac{m A^{3/2} K h}{\pi} \frac{H}{H}, \quad (6.2)$$

and the following conclusion on the “truncation” of the two-dimensional motion at high wave numbers can be drawn.

If a solution requires that a given proportion of the energy is to be represented explicitly, then the criterion on the grid length is

$$h = \frac{\pi}{m A^{3/2}} \frac{H \xi_g}{K}, \quad (6.3)$$

where $\xi_g \ll 1$ is a constant, chosen a priori.
For example, for an ocean 4 km deep, this criterion yields $h \sim 2 \times 10^8 \xi_g$. Thus, for a grid spacing of 20 km, 99% of the root-mean-square horizontal velocity is explicitly represented, with a proportionately lower percentage for a larger grid spacing.

Finally, substituting (6.2) in the frictional stresses (2.16), (2.17), we obtain operational relations

$$\frac{F_x}{\xi o} = \frac{K}{H} (1 + \xi^2) \ |Q| U,$$

$$\frac{F_y}{\xi o} = \frac{K}{H} (1 + \xi^2) \ |Q| V. \quad (6.5)$$

In a time-dependent solution, on the other hand, it is suggested that little of importance will be gained unless the time step, \(\Delta t\), is much less than the characteristic period of the turbulence, \(\chi\). This leads to the criterion for \(\Delta t\) that

$$\Delta t \ll m^{-1} A^{-3/2} H |Q|^{-1}. \quad (6.6)$$

This result is independent of grid length. Thus (6.6) ensures that the characteristic turbulence of the solution would be resolved regardless of the spatial scale of resolution.

The optimal choice of \(\Delta t\), however, is determined by the grid length. Clearly, the spatial scale of resolution, \(k_g\), should be related to the Nyquist frequency, \(\theta\), of the time-dependent solution by the equation

$$k_g = \frac{2\pi \theta}{|Q|}. \quad (6.7)$$

Substituting \(\theta = 1/(2 \Delta t)\) and using (6.1), we obtain

$$\Delta t = \frac{h}{|Q|} = \pi \xi_g \chi. \quad (6.8)$$

Eq. (6.8) always satisfies eq. (6.6), provided that \(\xi_g\) is sufficiently small. Thus the parameters of the time-dependent solution are determined by the two relations (6.3) and (6.8). For example, for a typical maximum oceanographic velocity, \(|Q_*| = 200 \text{ cm sec}^{-1}\) and \(H = 4 \text{ km}\), (6.6) yields \(\Delta t \ll 4 \text{ days}\); for \(h = 20 \text{ km}\), (6.8) yields the optimal time step \(\Delta t = 3 \text{ hours}\).

7. Discussion. It is now appropriate to examine the assumptions of the analysis in the real ocean. First consider the assumption of isotropy. The isotropic turbulent-energy equation (3.1) neglects local acceleration. This appears to be a reasonable approximation if
An estimate of the acceleration term is

\[ \frac{d}{dt} \frac{1}{2} \overline{u^2} \ll \varepsilon. \]  

(7.1)

Hence (7.1) gives the criterion for local isotropy:

\[ k_0 \lesssim mKH^{-1}; \]  

(7.2)

however, a precise definition of the inequality cannot be given without a detailed knowledge of the velocity field. Eq. (7.2) is probably satisfied in the interior of the ocean basin, where \( k_0 \sim L^{-1} \), and \( L^{-1} \) is a typical dimension of the basin. But in the region of narrow boundary currents, anisotropic effects are certainly important. Second, consider that, in the real ocean, dissipation probably occurs at all depths in the fluid, so that it is unrealistic to assume that all of the energy is transported to the bottom and dissipated there. However, as pointed out at the end of § 3, a generalization of the analysis along those lines would not lead to any fundamental differences. It appears generally that there is a separation between the characteristic period of the horizontal motions considered in this paper and the period of the eddies in the 3-D turbulence that give rise to the energy dissipation. These two frequency ranges are coupled by transitional quasihorizontal motions, the form of which is not specified by the above analysis. Third, consider that, for isotropic turbulence with an assumed distribution of horizontal velocities, the analysis could be extended for all \( \xi \), in the range \( 0 < \xi < \infty \), instead of for only the leading term, \( \xi \rightarrow 0 \), (cf. § 2). The benefits of the extension, in view of the above discussion, are equivocal.

Any live appreciation of these factors can be obtained only through experimentation either with numerical models having specified bottom-friction laws or in an ocean where measurements of current velocities are made systematically for extended periods.

The main conclusion of this analysis is that there does exist an energy-extraction range for 2-D isotropic turbulence in an ocean basin where the mean-squared vorticity is conserved and its magnitude is determined by the local horizontal velocity and the type of dissipation layer.
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