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Autospectra of Observed Particle Motions in Wind Waves

David H. Shonting

Department of Meteorology
Massachusetts Institute of Technology
Cambridge, Massachusetts

and

Oceanographic Branch
Naval Underwater Weapons Research and Engineering Station
Newport, Rhode Island

ABSTRACT

Autospectra of observed particle motions in ocean waves are presented. The spectra show the dominant peaks associated with the frequencies of the visually observed waves. Examination of the autospectra relative to depth reveals a strong attenuation of spectral energy, particularly at the higher frequencies; the peaks associated with the waves tend to redden the deeper the wave motions are observed. The time variability in the wave motions has been observed and correlated with changes in the wind field. Comparison of wave-motion spectra with free-surface spectra shows that they behave similarly and display an equilibrium range.

Introduction. A recent paper (Shonting 1967) has described Eulerian measurements of ocean waves obtained at Buzzards Bay Entrance Light Station (BBELS) with ducted impeller wave meters. This and an earlier paper (Shonting 1964) have described the use of this instrument to record one-dimensional or two-dimensional wave-motion components. Preliminary examination of field data has been presented; the variances and kinetic energy of the wind waves have been evaluated as a function of depth and as a function of various wave parameters. This study extends the data analysis of the observations to an examination of the autocovariance spectra of the time-series records.

This paper examines the autospectra of particle motions at various depths

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In various sea states. The spectra of the motions are compared with the 
spectra of the free surface-wave fluctuations obtained with a conventional 
wave-staff system.

**Instrumentation and Measurements.** The ducted-impeller wave meters and 
their calibration and the measurement procedures have been discussed (Shonting 1966, 1967). The signal output of the wave meter is in the form of voltage 
pulses whose repetition rate is directly proportional to the fluid velocity through 
the impeller (Shonting 1966). The response time of the impeller to step 
changes in water flow is of the order of 50 to 70 milliseconds. The threshold 
speed for impeller response (rotation) is about 5 to 7 cm sec^{-1}. Further analysis 
of the wave meter by Massey (1967) indicates that the response time is a 
function of the mean flow speed but that the instrument has a constant-
response distance of about 1 cm. Since the meter cylinder is 15 cm long, 
only orbital scales somewhat larger than 15 cm would be clearly defined. For 
wave-particle motions having speeds up to 200 cm sec^{-1}, the ducted meter 
effectively detects the velocity component parallel to the cylinder axis; i.e., 
the magnitude of the flow through the cylinder is approximately proportional 
to the cosine of the angle subtended by both the instantaneous direction of 
flow and the cylinder axis.

The wave observations were made from BBELS located 7 km off the 
southern coast of Massachusetts in 20 m of water. A detailed description of 
this platform has been presented (Shonting 1966). The wave meters were 
either mounted in a perpendicular manner to register horizontal and vertical 
velocity components at a given depth or fixed to a vertically suspended rod 
(with both axes vertical or horizontal) to record a particular component simult-
aneously at two depths.

**Data Processing and Spectral Analysis.** The method of abstracting the wave-
motion data from the strip-chart records is shown in Fig. 1. The top trace 
represents a record of the time variation in the 10-to-40-millivolt pulses 
generated by the impeller from a single meter. This record was processed on 
an electronic film reader, and the time of occurrence and the sign of each 
pulse were placed on punch cards. (The sign of each pulse denotes the flow 
direction through the meter.) These cards were then processed on a digital 
computer that formed a new series of “signed” time intervals (± ΔT_n). These 
new data were converted by means of the calibration information into a series 
of uninterpolated velocities, depicted by the middle plot in Fig. 1. The ve-
locities were then linearly interpolated at a time interval, ΔT, to produce 
the equispaced time series U_0, U_1, U_2... U_n. Since two channels of data 
were simultaneously recorded, a time-series pair was produced. The processed 
data were placed on punch cards and were also stored on magnetic tape for 
further analysis. For these observations, the range of frequencies to be observed
was centered around those associated with wind waves and swell. The typical wind waves observed at BBELS had frequencies ranging from 500 millicycles per second (mcps) to 200 mcps (2.0 to 5.0 sec period); the swell (when present) ranged from 150 mcps to 110 mcps (6.7-to-9-sec period).

The maximum frequency for which useful spectral information is obtained is the Nyquist frequency, \( f_N = (2 \Delta T)^{-1} \). If \( f_N \) is higher than the frequencies at which appreciable energy exists, "aliasing" can occur; high-frequency spectral energy accumulates in regions of lower frequency, and this results in distortion of the true spectrum (Kinsman 1965: 452).

The sampling rate, \( \Delta T \), for all wave records was 0.20 sec, giving a value for \( f_N \) of 2500 mcps. This frequency is well above those of wind waves. Since \( \Delta T \) is much larger than the observed response time of the impellers, velocity fluctuations equal to, or lower than, the Nyquist frequency should be faithfully recorded.

The record length, \( T \), was chosen sufficiently long to record from 60 to 100 cycles of the longest period waves (i.e., swell). Thus, sampling periods usually ranged from 5 min (300 sec) up to 10 to 15 min (600 to 900 sec); this insured proper sampling of even the long-period (9-to-10-sec) swell.

The time-series data were subjected to a spectral analysis similar to that
described by Blackman and Tukey (1959). For purposes of analysis, the measured wave-velocity component (the vertical component, \(w\), for example) is defined as

\[
w(t) = \bar{w} + w',
\]

where the barred term is the time average of \(w(t)\) for \(T\), and \(w'\) is the fluctuation about the mean.

The statistics of the fluctuating velocities such as \(w'\) are of chief interest, since these represent the velocity perturbations associated with the wave or turbulence motions. (In this study no distinction is made between wave and turbulence motions. The aim is to examine the statistics of all frequency contributions to \(w'\) or \(u'\) that were detected by our instrument.) The analysis involves calculation of the autocovariance function given by

\[
\varphi_w(\tau) = \frac{1}{2} \int_{-T/2}^{T/2} w'(t)w'(t + \tau)dt,
\]

where \(\tau\) is a particular time lag.

The autocovariance, being a function of the time lag, \(\tau\), is thus generated by displacement, multiplication, and integration of the argument of integral (2) for various values of \(\tau\). For a value of \(\tau = 0\), the function (2) becomes

\[
\varphi_w(0) = \frac{1}{T} \int_{-T/2}^{T/2} \bar{w}'^2 dt = \bar{w}'^2 = \sigma_w^2.
\]

This is the variance of \(w(t)\), which is proportional to the average kinetic energy of the wave motions at the depth of measurement (Shonting 1967).

The Fourier cosine transform of \(\Phi_w(\tau)\) provides the autocovariance spectrum (or simply autospectrum):

\[
\Phi_w(f) = \frac{2}{\pi} \int_0^\infty \varphi_w(\tau) \cos 2\pi ft d\tau.
\]

Taking the inverse transform of (4) and letting \(\tau = 0\), we have

\[
\varphi_w(0) = \int_0^\infty \Phi_w(f) df = \sigma_w^2.
\]

Practically speaking, since we are interested in a finite frequency band, integral (5) should be written

\[
\sigma_w^2 \approx \int_{f_L}^{f_N} \Phi_w(f) df.
\]

The highest frequency, \(f_N\), is determined by the criteria mentioned above. The lower frequency limit, \(f_L\), is the practical lower limit of the frequency
range from which useful information can be obtained. This frequency is
governed by $\tau$, chosen in eq. (2). According to Blackman and Tukey (1959: 11), it is not desirable to use lags longer than 5 to 10% of the total $T$.

The spectral analysis was done on a Control Data Corporation 3200 computer, using a modified version of the Tukey spectral-estimate program (Clark 1958). This program computes the functions (2) and (4) as truncated series expansions that utilize the time-series data (i.e., $U_0, U_1, U_2 - - - U_n$ in Fig. 1). This procedure used the Hanning Spectral Window to smooth the estimates (Blackman and Tukey 1958: 14).

The spectra were generated over 50 frequency bands, giving a frequency resolution of 50 mcps. At $\Delta T = 0.2$ sec the maximum $\tau$ is 10 sec; this exceeded even the longest swell periods. Also, the total lag time never exceeded 3 to 4% of the total sampling time.

It can be shown that, under certain assumptions, repeated sampling of the data provides an estimate of the spectrum, $\Phi_w(f)$, which is distributed approximately as chi-square for each spectral value. Thus, for each sample spectral estimate, the degrees of freedom were used to find the different confidence-limit bands for expected normal error distribution above and below the given value (Kinsman 1965: 448).

During each period of wave observation, the process measured ideally should remain stationary; i.e., the wave statistics should be unchanged during the sampling period. One cause of nonstationarity in the record is the existence of low-frequency components whose periods either fit into the observation period a small number of times or are even larger than the observation period (trend). Physical effects that induce nonstationarity in the wave records could be caused by rapid wind changes (hence by sea-state changes) and by relatively rapid fluctuations in the "mean" surface current. These components exert a complex distorting influence on the autocovariance function and the auto-spectra. Methods of removing nonstationary effects from wave data have been discussed by Sabinin and Shulepov (1965). Nonstationarity in real field data is difficult to observe except when a trend occurs. One way to estimate the degree of stationarity is to make a relatively long record, divide it into subsamples, and then examine their spectra. Comparison of the various spectra would serve to indicate the degree of stationarity in the long record.

During the series BBELS-11, a single 11-min (660-sec) record was made of the horizontal and vertical wave motions at 0.5 m beneath the mean wave-trough level. From the record, containing 3300 interpolated velocity values, two subsamples were analyzed; these consisted of the first 1000 data points (record A) and the last 1070 data points (record B). The two spectra are shown in Fig. 2. The ordinate, $\Phi_w(f)$, is the energy density in units of velocity squared per unit frequency ($\text{cm}^2 \text{sec}^{-1}$), and the abscissa is given as both frequency (mcps) and period (sec). Parameters describing the wind and sea conditions are shown in the graph.
BBELS-11
30 MARCH 1964
0·5 M

WIND
V ~ 12 M SEC\(^{-1}\), NW
SEA (FROM W, WNW)
HEIGHT ~ 120-140 CM
WAVELENGTH ~ 15-20 M

--- RECORD A
\[ \sigma^2_w = 1265 \text{ CM}^2 \text{ SEC}^{-2} \]

--- RECORD B
\[ \sigma^2_w = 1133 \text{ CM}^2 \text{ SEC}^{-2} \]

DF = 42

Figure 2. Autospectra from sections of an 11-min record.

The degrees of freedom for each calculated plot are about 42. The amplitude of the resulting 80\% confidence-limit-ratio obtained from a Chi-square graph is 0.74 below and 1.29 above the average value; it is indicated by a vertical bar.
By comparing the height of the bar with the amplitude of the spectral peak, we can assess their significance. The difference in the variances of the two records (i.e., the areas under the spectral curves) is about 10%. The overall shapes of the curves are similar, both displaying the dominant peak at 300 mcps (3.3-sec period). A small kink in both curves appears at about 100 to 150 mcps, and both spectra exhibit a similar downward slope associated with strong energy attenuation at higher frequencies. Some divergence of the curves occurs above 1300 mcps; however, this is in the region of the lowest spectral energy, and these differences contribute only slightly to the energy content of the two records. The strong similarities in the two curves suggest a reasonably stationary process for the duration of the 11-min record.

Examination of spectra from segments of the data together with the visual observations of the sea and wind conditions indicated that, in general, the 5-to-10-min sampling period of wave motions was unaffected by any slowly varying changes in the waves. Sudden wind shifts did occur, but these were unusual and were monitored as they happened.

**The Autospectra.** Four autospectra of the vertical wave-velocities particle associated with different wind and sea conditions are presented in Fig. 3. The spectra of the motions are very dependent upon the depth of the observed motion; hence, for comparison, the four observations were made one meter below the mean trough level of the waves. A linear ordinate scale is used in lieu of a log scale to show better the gross variations in the spectral density. The frequency scale was cut at an 800 mcps; beyond this frequency the spectral density is generally two orders of magnitude below the peak values. For comparison, the variances and wind velocities are listed on the graphs. Also given are values of $H$ (wave height) and $T$, estimated visually from the BBELS catwalk (some 17 m above the waves) by using a “seaman’s eye” and a stopwatch. Hence, the estimates must be considered extremely crude, and the values given only suggest the wave conditions present.

Observation 020 shows a rather broad spectrum, with the maximum occurring at about 150 mcps (6.7-sec period). The wind and seas were gentle, and a slight swell was radiating from the south. The spectral energy below 200 mcps probably is associated mostly with the swell. The record of 029 shows a very weak spectrum associated with light winds (about 1 m sec$^{-1}$) and very small waves.

Observations 061 and 081 represent spectra of relatively heavy seas, as indicated by the large variance (i.e., large kinetic energy) values and the high wind velocities. Note the double peaks in the 061 plot as compared with the single peak in 081, although the total energy is roughly equal. In 061 the waves were from the WNW, which is from the direction of the Massachusetts coast—a fetch limitation of about 7 to 10 km. The wind was steadily increasing in speed, so these wind waves were in the process of growing, were
relatively steep, and were of short period (3 to 4 sec). A few hours prior to the 061 observations, a NE storm had passed and generated large steep waves that radiated from the open sea to the SE. The seas at the time of the 061 measurements were confused, so it was difficult to separate visually the different wave directions. Although no swells were observed, they may have been
masked by the seas. The two peaks suggest that two dominant wave systems were present.

When the 081 observations were made, the SW wind had been blowing 6 to 8 hours, and the seas tended to be fully developed. No swell was observed. Thus, the 081 spectrum appears to portray fully developed, but locally generated, wind waves (sea). Most wave energy appears below 400 mcps (above 2.5 sec period). This cutoff may be caused by the instrument; but more probably it is a real effect, since the impellers easily responded to fluctuations above the Nyquist frequency of 2500 mcps.

The spectra in Fig. 3 show the correlation of the variance with the area
Figure 5. Spectra from two $w(t)$ records obtained simultaneously at 0.5-m and 2.0-m depths.

beneath the spectral curve [see eq. (6)]. The peaks of the spectra invariably occur at frequencies that coincide with those of the visually observed waves. This correlation was reported for earlier observations on wave-particle motion (Shonting 1964).

**Attenuation of Spectral Energy with Depth.** It has been demonstrated (Shonting 1967) that the variance of wind-wave particle velocities (or the wave kinetic energy) strongly attenuates with depth. This is displayed in Fig. 4,
which shows the variance of vertical velocities, \( \sigma^2_w \), as a function of depth for three series of observations in which the wind and sea conditions were similar. The values range from 1350 cm² sec⁻² immediately beneath the trough level of the waves to 37 cm² sec⁻² at a depth of 10 m. The rate of change in variance with depth is clearly of an exponential nature, as is indicated by the linear distribution of data values on the semilog plot. The energy decrease appears to fall off less rapidly below the 5-m and 6-m level.

It is of interest to examine the autospectra of wave-component motions as a function of depth. Fig. 5 shows two spectra of a record of the vertical velocity component measured simultaneously at depths of 0.5 m and 2.0 m. The
spectra show a dominant peak at 250 mcps (4-sec period), which is attributed
to seas produced by winds of \(10\) m sec\(^{-1}\). The 2.0-m spectrum (solid line)
shows a strong attenuation at all frequencies compared with the 0.5-m spectrum,
especially between 400 and 1500 mcps. The variance of the 0.5-m record is
1289 cm\(^2\) sec\(^{-2}\), but the variance of the 2.0-m record is only 624 cm\(^2\) sec\(^{-2}\).
The linear correlation coefficient for the two records is +0.924, showing an
expected strong in-phase relationship between the two velocities.

A series of records of vertical velocity motions were made at successive
depths (just beneath the trough level down to \(11\) m) from 1418 to 1618 on
7 June 1965. The winds were above 9.8 m sec\(^{-1}\) SSW and the observed wind
waves were 100 to 120 cm high and roughly 20 m long. The estimated
average period of the larger waves was about 4 sec. Six of the autospectra are
shown in Fig. 6. The variances (in cm\(^2\) sec\(^{-2}\)) are shown opposite the depths
of the measurements. The change in the areal content of the curves and in
the decreasing variances depicts the rapid decrease in energy with depth.

Of particular interest is a "red shift" in the peak frequencies as a function
of depth. The \(0\) m peak occurs at about 250 mcps, which was the average
frequency observed from the BBELS platform. The peak frequency decreases
with depth so that at about 7 m it has shifted to about 200 mcps (5-sec period)
and lies below the low-frequency peak attributable to ambient swells. Clearly,
the most attenuation of wind-wave energy occurs within the first 3 to 4 m of
depth. Below 9 to 10 m the motions at the frequencies of the wind waves are
virtually unobservable.

The Spectra of Wind-wave Buildup. Wave motions were observed while
the wind was steadily increasing. The vertical velocity component at 0.5 m
below the mean wave-trough level was observed twice on 26 January 1965—at
1145 to 1150 hours and at 1224 to 1229 hours. During the early morning
the winds were gentle at about 2 m sec\(^{-1}\) SE and the sea was relatively calm.
At about 1100 hours the wind began to increase steadily, and at 1224 the
wind velocity was 5.1 m sec\(^{-1}\) ESE. During this period the seas were in-
creasing in height, wavelength, and period. On this day a strong swell showed
no significant change during the course of the two records. The autospectra
of the two records are shown in Fig. 7 along with the variance values and the
environmental data. The degrees of freedom are 54 and 37 for the first and
second records, respectively.

The general shapes of the two curves are identical at all frequency ranges
except for the peaks (presumably associated with the wind waves) centered
at 650 mcps (1.5-sec period) in the first record and at 500 mcps (2-sec period)
in the second record. Furthermore, these peak values of spectral density in-
creased from 4.5 cm\(^2\) sec\(^{-1}\) to about 24 cm\(^2\) sec\(^{-1}\). The spectral peak attributed
to swell showed no significant change in the two sets of observations.

It is evident that the wind-wave growth is associated with the increased
magnitude of the physical parameters of the wind and waves and with the spectral peak that moved to a lower frequency and gained more energy (per unit bandwidth). The total wave energy likewise increased by about $26\%$, from $709\, \text{cm}^2\, \text{sec}^{-2}$ to $892\, \text{cm}^2\, \text{sec}^{-2}$. It appears that wind energy was imparted at the higher frequencies, with no evidence of interaction with the low-frequency swell.

It is of interest to compare this observed spectral change with similar observations. Bretschneider (1963) has presented a composite of successive free-surface autospectra records of the waves generated by Hurricane Donna as
she approached Atlantic City, New Jersey, on 12 September 1960. The recording device was a step-resistance wave gage located off the beach, where the depth was about 5 m. Bretschneider has shown the analog autospectra for four 20-min records obtained at 0000, 0400, 0800, and 1200 hours. Spectra 1 and 2 clearly show the swell peaks centered at about 0.11 to 0.13 cps (8–9-sec period) whereas the higher frequency wind-wave peaks show a buildup similar to that observed in the wave-motion spectra in Fig. 7. The wind-wave peaks grow in amplitude (i.e., energy) and shift to lower frequencies (i.e., reden) as a result of the approaching hurricane winds. The high seas soon dominated and completely enveloped the ambient swell (see Bretschneider 1963: curves 3, 4).

Recently Barnett and Wilkerson (1967) have reported results of wind-wave free-surface profile observations made with a modified airborne radar altimeter. The spectral peaks moved toward lower frequencies as the fetch from land increased. Thus, there seems to be a clear parallel in the time variability of both wave-particle-motion spectra and free-surface spectra as a function of wind increase and wave buildup.

The longest semicontinuous series of particle-motion observations was made over a 21-hour period to observe changes in wave statistics as the wind changed. A total of 39 individual time-series observations at various depths were made from 1622 hours on 29 March through 1251 hours on 30 March 1965. During this period a northeast storm system produced high winds that shifted gradually from ENE to WNW and then to WSW. These strong wind shifts, together with wind speeds ranging from 5 to 15 m sec⁻¹, generated a wide range of wave systems from which spectra could be observed. During the first 12 hours of observation, when the wind waves were from E and ENE out of Buzzards Bay, the wave meter was suspended from the east side of BBELS to avoid measuring waves on the leeward side of the tower. By 0400 hours on 30 March the wind had shifted to the WNW and the system was then moved to the west side of BBELS.

Nine wave observations of the vertical particle velocity, made at 0.5 m beneath the wave troughs, have been chosen as representative records for each phase of the changing weather conditions. The autospectra for this sequence are shown in Fig. 8. Greater resolution in the lower frequencies was obtained by computing these spectra with 100 lags (2) in lieu of the 50-lag resolution used otherwise. Also, a linear ordinate of spectral density has been used to emphasize the high-energy peaks. The time of sampling, the wind velocities, and the variances are shown for reference.

The first three spectra (030, 031, 032) display large variances in excess of 1000 cm² sec⁻². Strong bands of spectral energy occur from 150 to 350 mcps (12.9–6.6 sec), with peaks at about 250 mcps (4 sec). Observation 031 was only a 1-minute sample, hence there is a large uncertainty in the variance and spectral estimate.
Figure 8. Time variation in the auto spectra of $\omega$ during the passage of a northeast storm. 29–30 March 1965.

Even with relatively high ENE winds of 10 to 11 m sec$^{-1}$, the wave frequencies were concentrated at 250 mcps. Winds of 10 to 11 m sec$^{-1}$ from the S or SW (where there is an “infinite” fetch) produced waves having spectral peaks between 150 and 200 mcps (4.0 to 6.6 sec). The waves portrayed in spectra 030–032 were probably “fetch limited” because of the land masses
directly to the east of BBELS; therefore they display a relatively high frequency.

Observation 048 indicates a drop in wind speed, and the spectrum now shows two peaks: at 300 mcps (3.3 sec) and at about 200 mcps (5 sec). Here the variance value has decreased sharply to 686 cm$^2$ sec$^{-2}$. The lower frequency peak may be attributed to swell-like waves of longer period generated far from the local area. The spectrum of these waves could have been masked by the strong wind waves appearing in 030, 031, and 032.

Spectrum 049 shows a further decrease in wave energy associated with a lower wind speed. The low-frequency peak (200 mcps) seen in 048 is still present and relatively unchanged whereas the high-frequency peak (at 300 mcps) has decreased markedly. At the time of observation 052, the winds had dropped to 5.2 m sec$^{-1}$ and were more northerly; the high-frequency peak (still at about 300 mcps) is still smaller, and the low-frequency peak remains about the same as in 049. Comparing 048, 049 and 052, it appears that the energy between 250 and 500 mcps was rapidly attenuated as the wind decreased and shifted to a more northerly direction.

By 0440 hours on 30 March the wind was from the WNW at about 9.0 m sec$^{-1}$. The 053 spectrum shows that the low-frequency peak at about 200 mcps has been suppressed relative to the three previous spectra. The original band, centered at about 300 mcps, has vanished, and a new peak has formed at about 400 mcps. This latter peak is probably associated with the new wind waves generated by the freshening WNW winds.

The 054 autospectrum shows a further suppression of the low-frequency peak. There is, however, an increase in the height of the band associated with wind waves—now centered at about 350 mcps.

By 1140 hours the winds attained a speed of 14.0 m sec$^{-1}$. The large wind waves, which are commensurate with the large variance of 1133 cm$^2$ sec$^{-2}$, show up clearly (spectrum 057 B) as a strong spectral pedestal centered at about 300 mcps. There is a much smaller peak at 150 mcps. Actually, two trains of waves were visually observed, one from the NW, the other from the west. The latter waves were of lower frequency and had far less energy than the freshly developed waves related to the high winds from the WNW. Note that the larger peak in 057 B is similar to the peak in 032; both are attributed to relatively short-fetch wind waves generated by relatively high winds.

Comparison of the Spectra of Wave Motions and Free-surface Oscillations. It is of interest to compare the spectra of wave motions with similar statistics of the free-surface elevation, $\eta(t)$. The Coastal Engineering Research Center of the U.S. Army Corps of Engineers (CERC) has a step-resistance type of wave gauge mounted aboard the BBELS. This system records a continuous voltage analog of the free-surface function, $\eta(t)$, on a strip chart and on
magnetic tape recorders as well. These tapes are then analyzed on an analog computer. The tapes are run at 7500 times the recording speed, and bandpass filters are used to simulate the filtering at various wave frequencies. Comparison of autospectra obtained by analog and digital methods proved to be difficult. Specifically, a certain bias that appeared in the analog spectra was associated with the inherent nonlinearities of the filtering processes used. Also, comparison was made difficult because the CERC analyses filter out much of the spectral energy content above 1000 mcps (below a period on 1 sec). Some of the problems of comparing the digital with the analog-produced spectra have been discussed (Shonting 1967).

In spite of the difficulties, an effort was made to compare the wave-motion statistics with the free-surface data. The CERC wave records, occurring as a
strip-chart record of \( \eta(t) \) (in cm), were digitized at 0.2-sec intervals. The autospectra were computed in the same manner as the particle-velocity data. Fig. 9 shows spectra of the vertical motion measured at a depth of 3.0 m on 8 June 1965 between 1457 and 1501 hours; Fig. 9 also shows a free-surface record made on the same day from 1600 to 1610 hours. The autospectra, \( \Phi_w \) (cm² sec⁻¹) and \( \Phi_\eta \) (cm² sec), are plotted on the same ordinate scale.

The spectra up to 500 mcps (2-sec period) are very similar. From 500 to 1200 mcps the \( \Phi_w \) decreases much faster than the \( \Phi_\eta \) function. Since the \( w(t) \) record was obtained at a depth of 3.0 m, the accompanying attenuation of high frequencies with depth is undoubtedly reflected in the spectrum. Beyond this, \( \Phi_\eta \) seems to "whiten" and remain constant. The peaks at 250 mcps (4 sec) stand out, and even the small peak at 50 mcps is comparable in the two spectra. Since two different quantities are compared, not much can be concluded regarding the relative slopes of the two spectral densities.

A better comparison of the autospectra of the free-surface-wave records with the wave-particle-motion records could best be identified if a record of \( w(t) \), at the immediate surface, were available. This is because of the strong damping of the high-frequency surface motions with depth. Although the wave staff can detect the highest frequency motions (the limiting frequency being associated with the electronic response of the wave-staff detector and recorder), these motions are completely raw and unfiltered; on the other hand, the wave meter, lying beneath the free surface, detects motions that are subjected to natural low-pass filtering that attenuates the wave motions exponentially with depth. The long-period low-frequency motions do appear to be similarly represented by each spectrum in Fig. 9, indicating similarity between wave-particle-motion spectra and free-surface-motion spectra.

**Equilibrium Range of Wave Spectra.** Because of wave-energy inter-relationships, it has been suggested that the autospectra of the free-surface elevation, \( \eta(t) \), associated with wind waves should display, at some specific region of its spectrum, a functional relationship with frequency (Phillips 1958). This region of the spectrum is termed the "equilibrium range," because wave motions associated with frequencies in the range are saturated; i.e., they can hold no more wind-derived energy. Thus, there is a continuous outflow of spectral energy from this region. This saturation probably occurs because the physical characteristics of water limit the slope, height, and potential energy of waves of a particular frequency.

Consider the behavior of the energy spectrum, \( \Phi_\eta \), for relatively high frequencies. The physical parameters that govern both the behavior of the spectrum in the higher frequency ranges as well as the surface stability must be gravity (-g), wind speed, and some roughness parameter (e.g., the variance of the free-surface fluctuations) that is associated with the gross form drag of the waves. However, for the limiting stable configuration of the wave profile
(i.e., just before breaking occurs), the acceleration of a particle at the wave crest is approximately \(-g\). Using dimensional relationships, the equilibrium range spectrum has the form

\[ \Phi_\eta(f) = Ag^2 f^{-5}, \]  

(7)

where \(A\) is a constant.

It can be assumed that (7) holds for waves whose frequencies are somewhat lower than those of capillary waves, in which surface tension (in lieu of gravity) is the important restoring force (Kinsman 1965: 532). There is in the real waves, at any instant, a maximum in the spectral energy curve. Since the functional variation with \(f\) in (7) is monotonically decreasing, the relationship must obviously fail where spectral maxima occur. It is not clear just where the lower limit of the equilibrium range may be found; however, Pierson (1959) has offered evidence that, in sharply peaked spectra, the equilibrium range can begin only at frequencies greater than twice the frequency of the maximum spectral peak.

Workers have provided evidence that the slopes of \(\Phi_\eta(f)\) approximate the \(f^{-5}\) relationship indicated in (7). Burling (1955), who computed autospectra for 23 wave records obtained at Staines Reservoir, Middlesex, England, used a capacitance wave-staff recorder developed by Tucker and Charnock (1955). The wind speeds ranged from 5 to 8 m sec\(^{-1}\). The spectral analysis was done on a Fourier analyzer, described by Barber et al. (1946). Fig. 10 (from Burling

![Figure 10](image-url)  

Figure 10. The mean shape of the spectra at high frequencies weighted by \(f^n\); from Burling (1955).
1955) shows averaged values of the spectral density (weighted by various powers of frequency $f$) plotted against frequency. The various weighted values lie along almost straight lines. In the range of frequencies shown, the function $\Phi \eta f^n$ appears constant. Thus, it appears that, for frequencies from 850 to 1950 mcps (1.2-sec to 0.05-sec period), $\Phi \eta$ decreases nearly as the minus fifth power of the frequency.

Kinsman (1960) obtained a set of wave measurements in Chesapeake Bay that are not unlike those of Burling; he achieved a functional relationship wherein the high frequency side of $\Phi \eta$ varied approximately as $f^{-4.5}$. Barnett and Wilkerson (1967) also found an approximate fifth-power-law relationship in their airborne radar wave observations.

It has been demonstrated (Shonting 1967) that the observed wave motions are roughly similar to classical wave motions, at least with respect to approximate phase relationships of the $u$ and $w$ wave components. It is therefore reasonable to assume that the same forces that affect the spectral behavior of $\eta(t)$ likewise influence the motions $u(t)$ and $w(t)$. Moreover, since the value of $\eta(t)$ and of $u(t)$ and $w(t)$ are directly associated with the potential and kinetic wave energies (Shonting 1967), their spectra should be controlled by the same dynamic parameters.

According to Kinsman (1960), there should be a relatively constant decrease in spectral energy with increase in frequency in the region somewhat above the maximum peak energy. Also, it has been inferred that a saturated spectral region occurs irrespective of the absolute peak heights or wind speeds, assuming only that a minimum of wave energy is available to form the equilibrium range.

Examination of many of the particle-motion spectra has shown that, from 500 mcps (2 sec) through 800 mcps (1.25 sec), the slope of the autospectra functions was very similar and appeared to vary as $f^{-5}$. It was found that this similarity held even though the total spectral energy (variance) of each record varied widely and although the wind speeds associated with the waves varied from 5.2 m sec$^{-1}$ to 14.0 m sec$^{-1}$.

To explore further the functional relationship of $\Phi_w$ to $f$, several sets of some specific spectra of the vertical motions from BBELS-11 were examined. These observations were made at a depth of 0.5 m between 2009 hours on 29 March and 1137 hours on 30 March 1965. The winds throughout this period varied from 7.3 m sec$^{-1}$ ENE to 14 m sec$^{-1}$ WNW. Examination of the spectral plots showed a similarity in the values between 300 mcps and 1500 mcps and in the up-frequency slope of the function. Since the high-frequency region of the autospectrum was of primary interest, the values of $\Phi_w$ were averaged over 50 mcps increments from 300 to 1600 mcps. Weighted values of the spectral energy, using various powers of frequency, were then plotted as a function of frequency (Fig. 11) following the method of Burling (1955). The curve for $\Phi_w f^5 (n = 5)$ appears to be quite independent of frequency over the range from 400 mcps to 1400 mcps.
Figure 11. The average autospectra from BBELS-11 data weighted by $f^n$.

Conclusions. Statistical analysis of wave-particle-motion records obtained with ducted wave meters show that:

(i) Autospectra of the wave motions reveal peaks that correspond to the visually observed waves and contain magnitudes of energy commensurate with that associated with the wave field observed at the time of the observations.

(ii) Spectra of wave motions measured as a function of depth beneath the wave troughs display an exponential attenuation of wave energy. It is clearly shown that, with depth, the higher frequency energy is filtered out more rapidly than the lower frequency motions. Thus, a “red shift” occurs in the
dominant spectral peak when the wave motions are measured at increasingly greater depths.

(iii) Autospectra of wave motions that were measured over a time interval when the wind and sea state were increasing clearly show the increase in total spectral energy supplied by the wind. Also, it is shown that the energy peak associated with the wind waves moves down frequency as the wind waves build up and that the peak associated with the low-frequency swell did not change with time; this indicates that wind energy was only being fed into the higher (wind-wave) frequency region.

(iv) A 21-hour study of wave motions produced autospectra of several wave systems that are readily identified with the passage of a northeast-storm system. The various spectra of wave systems are associated with the time-variable wind speed, direction, and fetch. The wave kinetic energy, as indicated by the variance of the wave-motion component, is strongly related to the observed sea state and wind conditions.

(v) An autospectrum of the vertical wave-particle-motion component was compared with a free-surface-fluctuation record; both records show similar spectral peaks. The wave-motion spectrum shows the attenuation of higher frequencies not indicated on the free-surface record.

(vi) The high-frequency end of the particle-velocity autospectra appears to decrease as the minus-fifth power of the frequency similar to that of free-surface spectra. This suggests that the particle motions as well as the free-surface fluctuation of wind waves have associated with them an equilibrium range where wind-imported energy is continually shifted to other regions of the spectrum.

Further observations of wave motions with improved ducted meters are being made. It is hoped that other workers will undertake direct measurements of wave motions. By use of such devices as the ducted meters, much information can be gained about both the response of the sea-surface layers to wind stress and the generation of surface waves and currents.

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