A Model for Thermohaline Circulation in an Ocean of Finite Depth

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ABSTRACT

This paper presents a theoretical study of the steady-state density, pressure, and velocity fields in an unbounded ocean driven by the differential heating of the surface and by the divergence or convergence of the Ekman layer. An exponential dependence in the vertical direction for the density is assumed, and an exact solution that satisfies boundary conditions on a surface below the Ekman layer and on the ocean floor is given. For the case in which the characteristic depth of the thermocline is much less than the depth of the ocean, a relationship involving the characteristic depth, the east-west temperature gradient, the Ekman velocity, and the asymptotic vertical velocity is shown. The physical fields given by the solution are examined for an ocean of finite depth and are compared with those obtained when the thermocline is treated as a boundary layer.

1. Introduction. In recent years there have been several papers written about the oceanic thermocline and the thermally driven oceanic circulation (Robinson and Stommel 1959, Welander 1959, Robinson and Welander 1963, and Blandford 1965). These papers all treat the thermocline region as a boundary layer, at the bottom of which the temperature or density and the vertical-velocity component approach asymptotic values. Various boundary conditions have been applied for the top of the thermocline. This paper treats the thermocline in a manner similar to the approach used in most of the above papers, but it gives an exact solution that is valid to all depths and satisfies more general surface-boundary conditions than any previously given.

As in most of the above papers, we consider here the model of an ocean that covers a spherical earth. The depth of the ocean is taken to be much less than the radius of the earth, and the sum of the accelerations due to gravity and centrifugal force is assumed to be constant and directed radially. We consider a region of the ocean bounded above by a surface that is of constant radius and below the frictional Ekman layer. Since no vertical boundaries are

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imposed on the region and since friction is neglected, the solutions obtained are applicable to only the deep interior of a real ocean. The motion in the region is governed by the temperature distribution and vertical velocity at the upper surface. The vertical velocity at the surface is taken to be equal to that caused by the convergence or divergence of the Ekman layer and is referred to as the Ekman velocity.

In the region of the ocean considered, the effects of acceleration as well as those of friction are neglected in the equation that expresses conservation of momentum; the equation of state is linearized and includes the effects of pressure and temperature but not salinity; and, in the equation that expresses conservation of heat, the flux of heat by diffusion is taken to be in the vertical direction and is described by a constant eddy diffusivity. Using these equations, a single nonlinear differential equation for the pressure is derived. On the basis of certain assumptions, this equation is then written in a simplified form. Since the approximations that are made do not directly concern the behavior of the physical fields at great depth, solutions to the equation can be applied to all depths and are not restricted to a boundary layer, as in previous works.

In the remainder of the paper, similarity solutions for the equations are discussed with particular emphasis on the case in which the temperature at any position varies exponentially with depth. When the horizontal velocities also decrease exponentially at large depths, as in the boundary-layer case investigated previously, the solutions have noted similarities to those of both Welander (1959) and Blandford (1965). The solutions presented here, however, unlike those of Welander, include the effect of vertical mixing, and unlike those of Blandford, can fit an arbitrary surface-temperature condition with both a consistent thermocline depth and vertical velocity. For the boundary-layer case, a relationship involving the east-west temperature gradient, the Ekman velocity, the asymptotic vertical velocity, and a characteristic vertical scale is presented.

The properties of the solutions are also discussed for the case in which the normal velocity component at the bottom is taken to be zero. In this case, the form of the vertical velocity at the surface is altered, and for a given surface-temperature distribution, there may be added to the system a velocity field that has horizontal velocity components independent of depth and has a vertical velocity component at the surface that is restricted in form but is arbitrary in magnitude. Thus, some additional freedom in fitting the observed Ekman velocity is obtained.

2. The Basic Equations. In previous models of thermohaline circulation, the thermocline region has been treated as a boundary layer. The solution for the boundary-layer region then must be joined to a solution for a homogeneous deep layer. In this paper the thermocline and deep regions of the ocean are treated as one. In the deep ocean, where vertical changes in temperature are
relatively small while significant horizontal velocities may be present, the equations in Boussinesq form used for the boundary-layer models may not be valid. For this reason we derive the approximate equations used in this paper in such a way that it is easy to check \textit{a posteriori} whether the solutions obtained are consistent with the approximations made.

The effects of accelerations and friction are neglected in the conservation of momentum equations. For the thermocline region this may be justified in the same way as has been done for the boundary-layer models (Robinson and Welander 1963). For the deep ocean, accelerations may be neglected, since the velocities are even smaller than in the thermocline region. Friction is neglected on the assumption that the frictional boundary layer at the bottom is not important in determining the circulation in the interior of the ocean. The equations for the conservation of momentum, mass, and heat and for the linearized equation of state are

\begin{align}
2\mathbf{\Omega} \cdot \mathbf{\hat{v}} &= -\nabla P - g \rho \mathbf{k}, \\
\nabla \cdot (\rho \mathbf{\hat{v}}) &= 0, \\
\mathbf{\hat{v}} \cdot \nabla T &= G,
\end{align}

\begin{align}
\rho &= \rho_1 (1 - \alpha T + \beta P),
\end{align}

where $\mathbf{\Omega}$ is the rotation vector of the earth, $\mathbf{\hat{v}}$ the velocity vector, $P$ the pressure, $\rho$ the density, $\rho_1$ a reference density, $\mathbf{k}$ the unit vector normal to the surface of the sphere, $T$ the temperature, $G$ the divergence of the vertical flux of heat due to turbulent diffusion, $\alpha$ the thermal expansion, and $\beta$ the compressibility; $\alpha$ and $\beta$ are taken to be constant.

On taking the cross-product of (2.1) with $\nabla T$ and making use of (2.3), we have

\begin{align}
2(\mathbf{\Omega} \cdot \nabla) \rho \mathbf{\hat{v}} = \nabla T \cdot (\nabla P + g \rho \mathbf{k}) + 2 \rho \mathbf{\Omega} G.
\end{align}

On taking the divergence of (2.5) and making use of (2.2), we eliminate the velocity and obtain

\begin{align}
\{\nabla T \cdot (\nabla P + g \rho \mathbf{k}) + 2 \rho G \mathbf{\Omega}\} \cdot \nabla (\mathbf{\Omega} \cdot \nabla T) = \\
= (\mathbf{\Omega} \cdot \nabla T) \{2 \mathbf{\Omega} \cdot \nabla (\rho G) + g \nabla \rho \cdot (\nabla T \cdot \mathbf{k})\}.
\end{align}

Since, from (2.1),

\begin{align}
\mathbf{\Omega} \cdot \nabla P + g \rho (\mathbf{\Omega} \cdot \mathbf{k}) = 0,
\end{align}

$\rho$ can be expressed in terms of $P$ and, using (2.4), $T$ can also be expressed in terms of $P$. Thus, the differential eq. (2.6) can be given in terms of the pressure only. The solutions presented in this paper are for a simplified form of (2.6).
The simplified equation is written in terms of the surface coordinates $\theta$, $\varphi$, and $z$, where $\theta$ and $\varphi$ are the latitude and longitude (positive northward and eastward, respectively) and $z$ is the vertical coordinate (positive outward). The distance from the center of the earth to the level surface beneath the Ekman layer, taken to be constant, is denoted by $R$. In transforming expressions in vector notation into the coordinate system $\theta$, $\varphi$, $z$, terms of the order $z/R$ are neglected in comparison with terms of the order unity.

We now write

$$P = P_0 + P', \quad (2.8)$$

where

$$\nabla P_0 = -g \rho_1 (1 + \beta P_0) \hat{k}.$$ 

In addition, we assume that

(i) the horizontal derivatives of $P'$ may be neglected in comparison with vertical derivatives of the same order;

(ii) $g \beta \rho_1 P'$ and its derivatives may be neglected in comparison with $\partial P'/\partial z$ and its derivatives, respectively;

(iii) for $G = K \partial^2 T/\partial z^2$, where $K$ is the eddy diffusivity, we may replace $\rho G$ and $\nabla (\rho G)$ with $\rho_1 G$ and $\rho_1 \nabla G$, respectively.

Making use of these assumptions, we obtain from (2.6) the following approximate equation in $P'$ only:

$$P'_{zz} + \cot \theta P'_{zz} = \varepsilon \cos \theta \sin \theta \left( P'_{zz} - P'_{zz} \right), \quad (2.9)$$

where $\varepsilon = 2Q K \rho_1 R^2$ and $P'_\theta = \partial P'/\partial \theta$, $P'_z = \partial P'/\partial z$, etc.

To the same approximation, the temperature is given by

$$T = \frac{I}{\alpha \rho_1 G} P'_z, \quad (2.10)$$

and the velocity components from (2.5) are given by

$$u = -\frac{I}{2Q R \rho_1 \sin \theta} P'_\theta, \quad (2.11)$$

$$v = \frac{I}{2Q R \rho_1 \sin \theta \cos \theta} P'_\varphi, \quad (2.12)$$

$$w = \frac{I}{2Q R^2 \rho_1 \sin \theta \cos \theta} \left( \frac{P'_\theta P'_{zz} - P'_\varphi P'_{zz}}{P'_z} \right) + \frac{KP'_zz}{P'_zz}, \quad (2.13)$$
where \( u, \upsilon, \) and \( w \) are the velocity components in the directions of increasing \( \varphi, \theta, \) and \( z \), respectively.

In the remainder of the paper, certain solutions of (2.9) are examined; the arbitrary functions and constants in these solutions are determined from the following conditions:

(i) the temperature is given for the surface \( z = 0 \);
(ii) the vertical scale of the temperature field is given;
(iii) either the normal component of velocity at the bottom is zero or the horizontal velocity components approach zero at great depth;
(iv) the vertical velocity is given for the surface \( z = 0 \).

It is shown that not all of these conditions can necessarily be satisfied at the same time.

The separation of the pressure field into \( P_0 \) and \( P' \) has been carried out without reference to the physical situation. In fact, the term \( P_0 \) in the pressure is the usual unperturbed pressure for the state of no motion, with the density a function of pressure only. The term \( P' \) is the perturbation pressure for the fluid in motion, with temperature, \( T \), relative to the temperature for which \( \varrho = \varrho_1 (1 + \beta P_0) \). There is no intention of indicating that the three assumptions involving \( P' \) are basically different from those used by Robinson and Welander and others in deriving the equations in Boussinesq form for the boundary-layer case. In fact, the assumptions are essentially equivalent, and (2.9) for the perturbation pressure can be obtained from the approximate equations used by Robinson and Welander. However, it is shown that, in contrast to the thermocline region, for the deep ocean the first and second assumptions are only barely satisfied by the solutions obtained for \( P' \). It is of value, then, to have these assumptions stated in terms of \( P' \) so that it is possible to easily check a posteriori whether the solutions obtained are consistent with the approximations made.

In the remaining sections of the paper, the prime will be dropped from \( P' \) and we shall work with the perturbation pressure.

3. Similarity Solutions. Consider solutions of (2.9), which are separable through a similarity transformation of the form

\[
P(\theta, \varphi, z) = m(\theta, \varphi) G(\eta), \quad \eta = zk(\theta, \varphi).
\]

Substitution of (3.1) in (2.9) shows that \( m \) and \( k \) are given by

\[
m = \sin \theta^{i+j+2} \{ \varphi + E(\theta) \}^{2i+1}, \quad k = \sin \theta^i \{ \varphi + E(\theta) \}^i,
\]

where \( E(\theta) \) is an arbitrary function of \( \theta \) and where \( i \) and \( j \) are arbitrary constants. The function \( G \) satisfies
where the primes indicate differentiation by $\eta$.

If we equate the function $G$ used here with the first derivative of the function $F$ used by Robinson and Welander, (3.3) can be obtained from their equation for $F$. Using the values of $m$ and $k$ given by (3.2), it can be shown that their similarity solutions are also solutions of (2.9) when $P$ is given by (3.1). On the other hand, we cannot obtain their equation for $F$ from (3.3) without assuming that $G$ and its derivatives behave asymptotically in such a way that the horizontal velocity components go to zero at great depth. When this assumption is not fulfilled, (3.3) results in additional solutions for which horizontal velocities are present to all depths.

In general it is difficult to find a solution for (3.3) but, if $i = 0$ and $j = -1$, a solution is given by

$$G = a + be^{\eta},$$

(3.4)

where $a$, $b$, and $c$ are constants. When $a = 0$, this solution is equivalent to Blandford's exact solution for Robinson and Welander's equations. This solution gives a temperature field that varies exponentially with depth. By neglecting the vertical diffusion of heat, Welander (1959) obtained a solution for the problem of thermohaline circulation that also gives a perturbation density that varies exponentially with depth. We now consider, in general, solutions for (2.9) that result in density or temperature fields of this form. Since $P_z$ is proportional to the temperature, this is equivalent to looking for solutions of the form

$$P(\theta, \varphi, z) = A(\theta, \varphi) + m(\theta, \varphi) e^{z k(\theta, \varphi)}.$$  

(3.5)

Substitution of (3.5) in (2.9) leads to the conditions that, for nontrivial solutions,

$$k_\varphi = 0, \quad A_\varphi(k_\theta + \cot \theta k) = 0,$$

(3.6)

and

$$m_\varphi(k_\theta + \cot \theta k) = 0.$$  

For this system of equations there are two types of solutions:

(i) $k_\varphi = m_\varphi = A_\varphi = 0$, that is, $k$, $m$, and $A$ are arbitrary functions of $\theta$ only;

(ii) $k = c/\sin \theta$, where $c$ is an arbitrary constant, and $m$ and $A$ are arbitrary functions of $\theta$ and $\varphi$.

In order to treat the thermocline region as a boundary layer and have the horizontal velocity components approach zero at great depth, we must take $A = 0$. Then solutions of the form (3.5) correspond to solutions of Robinson and Welander's $M$-equation

$$M = -2\pi^2 \sin^2 \theta k_1(\theta, \varphi) \varphi + m_1(\theta, \varphi) e^{z k_1(\theta, \varphi)} + C(\theta),$$

(25, 3)
where \( C \) is an arbitrary function and where \( m_1 \) and \( k_1 \) satisfy eqs. (3.6) for \( m \) and \( k \). For arbitrary \( m_1 \), solutions of this form cannot be written in the similarity form used by Robinson and Welander; in general they are distinctly different. Blandford's exact solution, however, is a common particular case for these solutions as well as for Robinson and Welander's similarity solutions.

4. Solutions of type (i). For these solutions, the pressure field is given by

\[
P(\theta, z) = A(\theta) + m(\theta) e^{z k(\theta)};
\]

the temperature from (2.10) is given by

\[
T(\theta, z) = \frac{km}{\alpha g \theta_1} e^{kz};
\]

and the velocity components from (2.11) to (2.13) are given by

\[
u(\theta, z) = 0,
\]

\[
w(\theta, z) = Kk.
\]

Since, on the surface \( z = 0 \), the temperature is proportional to \( km \) and the vertical velocity to \( k \), we may choose any north-south dependence for the temperature and vertical velocity on this surface. Also, since, for a realistic thermocline, the magnitude of \( k \) must be of the order of \( 10^{-5} \) cm\(^{-1}\), the vertical velocity is of the order of \( 10^{-5} \) cm/sec, for \( K \) of the order of \( 1 \) cm\(^2\)/sec. These values are consistent with realistic oceanic values; however, since this solution results in physical fields that vary only with latitude, this solution is of little value in describing the real ocean and will not be discussed further.

5. Solutions of Type (ii). For these solutions the pressure field is given by

\[
P(\theta, \varphi, z) = A(\theta, \varphi) + m(\theta, \varphi) e^{cz/\sin \theta};
\]

the temperature from (2.10) by

\[
T(\theta, \varphi, z) = \frac{cm}{\alpha g \theta_1} \frac{e^{cz/\sin \theta}}{\sin \theta};
\]

and the velocity components from (2.11) to (2.13) by

\[
u(\theta, \varphi, z) = -\frac{1}{2 \Omega R \theta_1 \sin \theta} \left\{ A_0 + \left( m_0 + czm \frac{\cos \theta}{\sin^2 \theta} \right) e^{cz/\sin \theta} \right\};
\]

\[
w(\theta, \varphi, z) = Kk.
\]
\[ v(\theta, \varphi, z) = \frac{1}{2\Omega R_0 \sin \theta \cos \theta} \left\{ A_\varphi + m_\varphi e^{cz/\sin \theta} \right\}, \quad (5.4) \]

\[ w(\theta, \varphi, z) = \frac{1}{2\Omega R^2 \varphi_1 \sin \theta} \left\{ \frac{m_\varphi e^{cz/\sin \theta}}{c \sin \theta} + \frac{A_\varphi m_\varphi - A_\varphi m_\theta}{cm \cos \theta} \right\} + \frac{A_\varphi}{c \sin \theta} + \frac{A_\varphi c_2}{\sin^2 \theta} \left( \frac{Kc}{\sin \theta} \right) + \left( \frac{c_2}{\sin \theta} \right). \quad (5.5) \]

Since \( m \) is an arbitrary function of \( \theta \) and \( \varphi \), the observed surface temperature can be exactly described by (5.2). The model gives a local exponential variation in temperature with depth, with a vertical scale of \( \sin \theta/c \). The value of \( c \) can be chosen to give the correct vertical scale at only one latitude. Whether the choice of \( m \) and \( c \) can give a temperature field that agrees with observation can be checked with a meridional temperature profile, such as that in Fig. 1; in this case the surface temperature is proportional to \( \cos (\theta + 10^\circ) \), the reference temperature is \( 2.45^\circ \text{C} \), and \( c = (1500 \text{ m})^{-1} \). This surface-temperature distribution and the resulting temperature field at depth show qualitative agreement with the temperature field for the west side of the ocean as presented by Robinson and Stommel (1959: fig. 2).

Let us now consider the velocity fields for the boundary-layer case in which \( A = 0 \) and the horizontal velocity components approach zero at great depth. When \( A = 0 \), (5.5) for the vertical velocity reduces to

\[ w = \frac{m_\varphi e^{cz/\sin \theta}}{2\Omega R^2 \varphi_1 \sin \theta} + \frac{Kc}{\sin \theta}. \quad (5.6) \]

![Figure 1. North-south section of the temperature field, with the surface temperature proportional to \( \cos (\theta + 10^\circ) \) and \( c = (1500 \text{ m})^{-1} \).](image-url)
The second term on the right side of (5.6) is independent of depth and represents the asymptotic vertical velocity, \( w_\infty \). With the same value of \( \epsilon \) as that used previously—that is, \((1500 \text{ m})^{-1}\)—and \( K = 1 \text{ cm}^2/\text{sec} \), \( w_\infty = 1.3 \times 10^{-3} \text{ cm/sec} \) at a latitude of \( 30^\circ \). Since such a value roughly corresponds to what is usually accepted for the oceans and since \( K \) is at best known within an order of magnitude, this is consistent with observation.

Let us now consider the first term on the right side of (5.6), which we denote by \( w_1 \). At great depth this term is negligible, and at the surface \( z = 0 \) is equal to the difference between the Ekman velocity, \( w_E \), and \( w_\infty \). It follows from (5.2) and (5.6) that

\[ T_\phi = \frac{2 \Omega R^2 \epsilon^2}{\alpha g} w_1, \quad (5.7) \]

which, at the surface \( z = 0 \), takes the form

\[ T_\phi = \frac{2 \Omega R^2 \epsilon^2}{\alpha g} \{ w_E - w_\infty \}. \quad (5.8) \]

Since the characteristic vertical scale determines the value of \( \epsilon \) and thus of \( w_\infty \), we see from (5.8) that there is a definite relationship for these solutions between the vertical scale, the Ekman velocity, and the zonal temperature gradient at the surface \( z = 0 \). Computations by Montgomery (1936) for \( 30^\circ \text{N} \) in the Atlantic show that the convergent Ekman layer results in a downward vertical velocity of about \( 10^{-4} \text{ cm/sec} \). If we take \( \Omega = .7 \times 10^{-4} \text{ sec}^{-1} \), \( R = 6.3 \times 10^8 \text{ cm} \), \( \alpha = 2.0 \times 10^{-4} \text{ C}^{-1} \), \( g = 981 \text{ cm/sec} \), \( K = 1 \text{ cm}^2/\text{sec} \), and the value of \( \epsilon \) as chosen above, we obtain from (5.8) \( T_\phi = -1.4 \text{ C/rad} \) at \( 30^\circ \text{N} \). This is not much less than the observed zonal gradient; however, not too much weight should be given to this agreement because of the probable inaccuracies in the values of \( \epsilon \), \( K \), and \( w_E \).

It is perhaps more important to see whether (5.8) holds qualitatively. Robinson and Stommel (1959: fig. 2) have presented the surface temperature distribution for the North Atlantic and have shown that \( T_\phi \) varies from a value of about \(-5 \text{ C}/\text{rad} \) at \( 20^\circ \text{N} \) to \( 0 \text{ C}/\text{rad} \) at about \( 50^\circ \text{N} \) and to positive values farther north. Montgomery's data show that \( w_E \) varies from a value of about \(-15 \times 10^{-5} \text{ cm/sec} \) at \( 20^\circ \text{N} \) to a zero value at about \( 45^\circ \text{N} \) and to positive values farther north. These values of the Ekman velocity and temperature gradient vary qualitatively in the way predicted by (5.8) so long as the magnitude of \( w_E \) over most of the region is somewhat greater than \( w_\infty \). For the values of \( \epsilon \) and \( K \) given above, this is the case in the region of the North Atlantic considered.

It is now apparent that, unlike Blandford's exact solution, the solution presented here for the boundary-layer model is able to fit an arbitrary surface temperature; in addition, for a reasonable vertical scale, it gives a vertical
velocity that agrees well with values inferred from observation. We now con-
sider the modifications to the theory when we take $A(\theta, \varphi) \neq 0$ and obtain
nonzero horizontal velocities at the bottom of an ocean of depth $h$. At the
ocean's bottom we require that the component of the velocity normal to the
bottom be zero. For the simple case of $h$ constant, we obtain from (5.5)

$$w = \frac{1}{2 \Omega R^2 Q_1} \left\{ \frac{m_\varphi}{c \sin \theta} \left( e^{cz/sin \theta} - e^{-ch/sin \theta} \right) + \frac{A_\varphi(z + h)}{\sin^2 \theta} \right\}, \quad (5.9)$$

where the function $A$ satisfies

$$\frac{m_\varphi}{c \sin \theta} e^{-ch/sin \theta} + \frac{A_\theta m_\varphi - A_\varphi m_\theta}{cm \cos \theta} + \frac{A_\varphi + \delta c^2}{c \sin \theta} - \frac{A_\varphi h}{\sin^2 \theta} = 0, \quad (5.10)$$

wherein $\delta = 2 \Omega R^2 Q_1 K$.

A particular solution to (5.10) has not been found for arbitrary $m$. How-
ever, to any particular solution may be added solutions of the form $\Phi(m e^{-ch/sin \theta}/sin \theta)$, where $\Phi$ is an arbitrary function and the quantity $m e^{-ch/sin \theta}/sin \theta$ is
proportional to the temperature at the bottom, $T_B$. The arbitrary function
$\Phi$ may be used to help fit the Ekman velocity at the surface through the con-
tribution of $\Phi_\varphi$ to $A_\varphi$.

In general, it is feasible to find only a numerical solution to (5.10). The
degree to which the numerical solution can satisfy boundary conditions on
the boundaries of the region over which the equation is integrated depends
on how well the arbitrary function, $\Phi$ of $T_B$, can be employed to fit the
boundary conditions. When the region considered is bounded by two lines of
latitude and two of longitude and when $m$ has values consistent with the
temperature distribution in the North Atlantic, the value of $A$ can be fixed
on one of the lines of latitude and on one of the lines of longitude. The solu-
tion so obtained may be modified by any arbitrary function of $T_B$ in such a
way as to give the best fit to the observed Ekman velocity over the entire
region.

A particular solution of some limited practical value can be obtained when
$m_\varphi$ is independent of $\varphi$. In this case the general solution may be written:

$$A = -\frac{m e^{-ch/sin \theta}}{sin \theta} \left\{ \delta c^2 \int_0^{\cos \theta} e^{ch/sin \theta} m_\varphi d\theta + sin \theta \right\} + \Phi(m e^{-ch/sin \theta}/sin \theta). \quad (5.11)$$

If $T_\varphi$ and thus $m_\varphi$ goes to zero at some latitude, $\theta_o$, the $\theta$ derivative of the
first term on the right side of (5.11) is divergent at $\theta = \theta_o$. For at least some
forms of $m_\varphi$, the arbitrary function $\Phi$ can be chosen so as to keep $A_\theta$ finite
at $\theta = \theta_o$. Thus, the solution does not necessarily predict an infinite zonal
current at the latitude where $T_\varphi$ is zero. Because (i) the behavior of the solu-
tion near $\theta_o$ depends on our particular choice of $m_\varphi$ as independent of $\varphi$ and
on the form of $m_\varphi$ near $\theta_o$, and, because (ii) the model does not include the
effects of horizontal friction and accelerations, no detailed investigation of the
solution near $\theta = \theta_o$ is warranted. In the North Atlantic, $T_\varphi$ is zero on the
average at about $50^\circ$N, which is somewhat north of the region where the
Gulf Stream crosses the ocean. In the region between about $10^\circ$N and $40^\circ$N,
the solution given by (5.11) should be valid within the limitations imposed
by the general theory and by the choice of $m_\varphi$ as independent of $\varphi$.

An approximate particular solution to (5.10) for large values of $h$ is given by

$$A^P = \frac{\delta c}{h} \sin \varphi - me^{-ch/\sin \theta}.$$  \hspace{1cm} (5.12)

For realistic values of $m$, $c$, and $\delta$, this solution neglects terms of the order
$\sin \theta/ch$ compared with those retained. Since the only approximation is involved
in the term proportional to $\delta$, $A^P$ is an exact particular solution in the limit of
small $\delta$ or $K$ as well as in the limit of large $h$.

Since $A^P \to 0$ for large $h$, we obtain with $A = A^P$ the boundary-layer
solution in the limit of large $h$. Thus the boundary-layer solution is a valid
limit to a particular solution for an ocean of finite depth. Previous workers
have considered that the solution for the boundary layer must be joined to
the solution for a homogeneous deep layer. In § 6, where $h$ is taken to be
5000 m, the velocity fields and transports obtained from such an approach
are compared with those obtained when the ocean is treated as one layer,
with $A = A^P$. We shall neglect terms in $A$ of the order $\sin \theta/ch$, which has a
value of 0.15 at $30^\circ$N for $h = 5000$ m and $c$ as chosen previously.

6. Velocity Fields and Transports. With $A = A^P$ for the one-layer ocean,
the velocity field near the surface $z = 0$ approaches that for the boundary-
layer solution for large $h$. When $h = 5000$ m, the horizontal velocity com-
ponents at the surface for both cases depend almost entirely on the terms in
(5.3) and (5.4), which contain $m$ and its derivatives. The only differences,
arising from the contribution of the derivatives of $A^P$, are of the order $\sin \theta$
$w_c/chw_c$, which, based on previous estimates, is about equal to 0.02 at $30^\circ$N.

At the bottom of the one-layer ocean the horizontal velocity components
depend almost entirely on the derivatives of $A^P$ and are approximately given by
$u = -RKc\varphi \cot \theta/h$ and $v = RKc/h \cos \theta$. For the boundary-layer case,
the velocity field in the homogeneous deep layer is determined by the following
requirement: the vertical velocity at the bottom must be zero and at the sur-
face must be equal to the asymptotic value of the vertical velocity for the
boundary layer—that is, $Kc/\sin \theta$. The resulting horizontal velocity com-
ponents differ from those given for the bottom of a one-layer ocean by the
factor $h/h'$, where $h'$ is the depth of the homogeneous deep layer$^2$. Since in

2. This does not include the arbitrary part of $u$ that is independent of longitude.
realism the depth of the boundary layer is a considerable fraction of the ocean's depth, \( h \), the factor \( h/h' \) is somewhat different from unity and the horizontal velocities at the bottom for the two cases are significantly different. These differences are greater in the midocean region, where the terms in (5.3) and (5.4) that depend on the derivatives of \( A \) and those that depend on \( m \) and its derivatives have similar magnitudes.

For the two cases, the main portion of the transports, differing only by terms of the order \( e^{-ch\sin \theta} \), arises from the part of the horizontal velocity components that is proportional to \( m \) and its derivatives. This portion of the transport is an order of magnitude greater than the remainder of the transport, which is identical for the two cases and arises from the part of the horizontal velocity components that is proportional to \( \delta \) or \( K \). For both cases the ratio of the zonal-to-meridional-transport is given approximately by \( \cos \theta (T_s \sin^2 \theta)_{Q} / (T_s \sin^2 \theta)_{QP} \), where \( T_s \) is the surface temperature. For the surface-temperature distribution used previously, this ratio is approximately 7:1 at 30°N. The direction of the total transport is slightly to the south of west.

For an ocean of 5000-m depth with the near-surface distribution of properties considered above, it is now apparent that the treatment of the ocean as a boundary layer over a deep homogeneous layer results in essentially the same velocities and transports as those obtained for a one-layer ocean with \( A = A_P \). The only significant differences exist in the deep horizontal velocity components. We now consider the additional velocity field, which is obtained from the solution presented in this paper when the arbitrary function, \( \Phi \), is included in \( A \).

It has been shown that the first term of (5.6) or (5.9) gives reasonable agreement with the observed Ekman velocity in the North Atlantic. To obtain an equal contribution to the vertical velocity at the surface, the magnitude of \( \Phi \) must be \( \sin \theta m_P / ch \). It follows from (5.4) that this value of \( \Phi \) contributes significantly to the meridional velocity and gives a meridional transport of magnitude \( m_P / 2 R \Omega \theta_{Q} \cos \theta \). This is approximately equal to the meridional transport from all the other terms considered previously. Since \( \Phi_{\theta} = \Phi \sqrt{(T_B \theta)/(T_B \theta P)} \), the corresponding zonal transport can be obtained from (5.3). It is about a factor \( c h T_s \cot^2 \theta / (T_s \theta P \) greater than the meridional transport. For the surface-temperature distribution in the North Atlantic used previously, this factor has a value of about 80 at 30°N. Thus, the total transport due to \( \Phi \) is almost zonal. The direction of the transport depends on whether \( \Phi \) gives a positive or negative contribution to the vertical velocity at the surface.

The above estimate gives a zonal transport due to \( \Phi_{\theta} \) that is some ten times greater than that arising from the other terms. However, we have chosen \( \Phi \) to obtain a contribution to the vertical velocity at the surface that is equal to that of the first term in (5.9). Since in practice it may be necessary to only choose \( \Phi \) to obtain a small correction in this term, the zonal transport due
to $\Phi_0$ may actually be of the same order as that arising from other terms. In that event the contributions of $\Phi_p$ and $A_p^P$ to the vertical velocity at the surface will be of the same magnitude.

7. Discussion. Let us now examine the validity of the assumption made in the derivation of (2.9), that horizontal derivatives of the pressure may be neglected in comparison with vertical derivatives of the same order. It follows from (5.1) that

$$\frac{1}{R} \frac{P_p}{P_z} = \frac{\sin \theta (A_p + m_p e^{cz / \sin \theta})}{e R m e^{cz / \sin \theta}} \quad (7.1)$$

Taking for $A_p$, the maximum value $m_p \sin \theta / ch$, indicated above, and taking $m$ values consistent with the temperature distribution in the North Atlantic at $30^\circ\text{N}$, the value for the right side of (7.1) is about $10^{-5}$ at the surface and $2 \times 10^{-4}$ at 5000 m. Similarly, the ratio of $(1/R) P_0$ to $P_z$ is about $3 \times 10^{-5}$ at the surface but, because the magnitude of $\Phi_0$ is greater than that of $\Phi_p$, the ratio is about $10^{-1}$ at 5000 m. A similar estimate for the second assumption concerning the terms involving $\beta$ shows that at $30^\circ\text{N}$ the magnitude of the largest terms neglected is at most $10^{-1}$ to $10^{-2}$ times the magnitude of those retained. Since both of these estimates are based on the maximum indicated value of the contribution of $\Phi_p$ to the Ekman velocity, the ratio of the neglected terms to those retained may be another order of magnitude smaller. In general, however, because the ratio depends critically on the latitude and on the magnitudes of $c$, $h$, and the derivatives of the function $\Phi$, it should be calculated \textit{a posteriori} if the solution is applied to any other region of the world's oceans.

Because of the way in which some of the neglected terms depend on the latitude and on the derivatives of $\Phi$, in some region near the equator $\Phi_p$ must be made so small that it cannot be used to fit the Ekman velocity. Of course, within a few degrees of the equator the model is not valid, since the effects of accelerations and friction have been neglected.

It is now evident that the solution given is a valid approximate solution to (2.1)–(2.4) and, in the region of the North Atlantic considered, the solution can fit the observed surface-temperature distribution exactly, can fit the observed vertical scale, and, to some degree, can fit the observed Ekman velocity. The model, of course, is restricted in its applications to the oceans by the nature of the basic equations which, for example, give the density as a linear function of temperature and pressure. Thus, the salinity has not been included in the equation of state, and the equation expressing conservation of salt has been neglected. This deficiency may be partially overcome by introducing an apparent temperature that depends linearly on both salinity and temperature.
Assuming a linear equation of state and equal eddy diffusivities for heat and salt, the basic equations take the same form as (2.1) to (2.4), with the apparent temperature replacing $T$. The apparent temperature must then be fitted to the observed data. In the region of the North Atlantic considered, where the density is primarily determined by the temperature and pressure, use of an apparent temperature is not important in discussing the qualitative agreement of the solution with observed data. If, however, we wish to obtain better agreement with the details of the density distribution or to compute the velocity fields, the apparent temperature should be used. In other regions of the world's oceans this may be still more important.

Another limitation in the model as presented arises from the use of a level bottom for the ocean. Inclusion of an uneven bottom increases the complexity of the coefficients in the first-order partial differential equation for $A$; however, since this equation in practice must be solved numerically, this leads to no great additional effort in the solution of the equation and in the computation of the velocity components.

Finally, the model is limited by the exponential variation in the perturbation temperature with depth. Although this form of the temperature agrees roughly with observation and enables us to fit the observed surface temperature exactly, it limits us in the forms of the Ekman velocity that can be fitted. In the North Atlantic it appears that this restriction is not important in obtaining qualitative agreement of the model with observed data. For other areas of the world's oceans it may be necessary to obtain solutions with different dependence on the depth in order to obtain agreement with the observed data.

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