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Tsunami Response for Islands: Verification of a Numerical Procedure

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ABSTRACT

A numerical integration procedure is employed to evaluate the response of the water level in the near field of an island due to the incidence of plane monochromatic gravity waves that have periods in the tsunami range. Verification of the numerical procedure is restricted to two axially symmetric islands for which analytical solutions are available. The first case is that of a vertical cylinder of circular cross section in water of constant depth. The second case employs a depth variation near the island that is proportional to the square of the radial distance out to a prescribed distance, beyond which the depth is constant. The boundary condition employed at the islands is that of no radial flow. A radiation condition is used in the far field for the scattered waves. Comparison of the numerical and analytical solutions for the amplitude and phase of the resulting waves around the perimeter of the island indicates that, for the periods investigated, the numerical procedure reproduces the analytical results quite faithfully.

Introduction. The tsunami phenomenon may be considered in four general phases: generation, propagation and dispersion, topographic modification, and runup. The modification and reflection of a tsunami occur during the advancement of waves from deep water into the more shallow water above the continental shelves or around anomalous features such as islands or seamounts that extend upward from the depths. Little is understood about the relative magnitudes of the effects of modification mechanisms, although preliminary studies of tsunami-island scattering interaction have been made. Examination of the scattering contribution to tsunami modification has developed analytically for simple boundary geometry. Independent investigations by Omer and Hall (1949) and Hidaka and Hikosaka (1949) have represented the island of Kauai in the Hawaiian chain as a circular cylinder (Fig. 1) in water of constant depth. Adapting some results from acoustic theory, qualitative agree-

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ment has been found between the theoretical shoreline response pattern set up by monochromatic incident waves and the smoothed runup heights taken from field data obtained during the tsunami of April 1, 1946. Homma (1950) has introduced a further refinement by making allowance for a paraboloidal bathymetry out to a prescribed distance, beyond which the depth remains constant. His studies have indicated that scattering can substantially alter long waves in the near field. However, the constraint imposed by the simple geometric island shapes has limited the interpretation of the analytic results and has precluded a thorough investigation into the role of scattering and diffraction.

The work reported in this paper precedes an attempt to extend the study of scattering to a numerical experiment in which the given bathymetry and island configuration are more precisely represented. The development of the numerical experiment has proceeded with an adaptation of the classical linearized long-wave equation and appropriate equivalents of the boundary conditions. The verification of this numerical analogue is achieved by comparison of the computer solutions for cylindrical and paraboloidal islands with known analytical solutions. It is the adequacy of the numerical approximations of the boundary conditions that we consider to be of greatest interest, particularly the far-field condition.

**The Boundary-value Problem.** The analytic solutions, which serve as the standards for comparison with the numerical method, are selected from the general solutions of the polar \((r, \theta)\) form of the linearized long-wave equation

\[
\frac{\partial^2 \zeta}{\partial r^2} = g \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( rD \frac{\partial \zeta}{\partial r} \right) + D \frac{\partial^2 \zeta}{\partial \theta^2} \right].
\]

(1)

In this relationship, \(\zeta\) is the elevation of the free surface above mean water level, \(g\) is the acceleration of gravity, and \(D\) is the depth, measured from mean water level, which may vary radially. The analytic form of the general solutions of (1) is governed by the radial dependency of the depth. The specification of a unique solution requires the application of the inner and outer boundary conditions. At the shoreline \((r = r_o)\), the two geometric islands considered here extend vertically through the sea surface as impermeable fixed circular cylinders. In this case the boundary condition appropriate to the phenomenon requires zero component of radial flow, or in terms of \(\zeta\):

\[
\frac{\partial \zeta}{\partial r} = 0.
\]

(2)

The outer boundary condition is a condition of radiation for scattered waves. It is a statement that the energy contained in the scattered portion of the \(\zeta\) field, \(\zeta_s\), is progressing radially outward to infinity in the far field. Such a
radiational condition has been given by Sommerfeld (1949) in the differential form
\[
\frac{\partial \zeta_s}{\partial t} + (gD)^{1/2} \frac{\partial \zeta_s}{\partial r} \to 0
\]  
(3)
as \( r \to \infty \), with \( D \) assumed to be constant beyond some finite value of \( r \). For incident monochromatic plane waves having surface displacement
\[
\eta = e^{i(kr \cos \theta - \omega t)}
\]  
(4)with wave number \( k \) and angular frequency \( \omega \), condition (3) may be restated as a requirement that the \( \zeta \) field approach the form
\[
\zeta = \eta + F(\theta) r^{-1/2} e^{i(kr - \omega t)}
\]  
(5)as \( r \to \infty \). The second term in (5) represents the far-field, outwardly progressing, scattered waves that have an angular beam pattern \( F(\theta) \). The analytic form of (5) is derived from the asymptotic far-field approximation of the Hankel function solutions of (1) for water of constant depth.

The solutions for boundary-value problems of this nature are well known and documented in standard texts, such as that of Morse and Feshbach (1953). The application of the boundary conditions to general solutions of (1) leads to the following relationships for the diffraction patterns at the shorelines.

[i] For the cylindrical island (Fig. 1) in water of constant depth,
\[
\zeta(r_0, \theta, t) = \frac{2}{\pi kr_0} \sum_{n=0}^{\infty} i^{n+1} \frac{\epsilon_n \cos n\theta}{H'_n(kr_0)} e^{-i\omega t},
\]  
(6)where \( H_n \) refers to the \( n \)th-order Hankel function of the first kind \( (H_n = \mathcal{H}_n + i \mathcal{Y}_n) \), \( \epsilon_n \) assumes the value 1 for \( n = 0 \) and the value 2 for \( n \neq 0 \), and the prime indicates a differentiation of the function with respect to its argument. The amplitude, \( A(\theta) \), and the phase,\( ^2 \phi(\theta) \), of the wave pattern are given by

2. Comparison with (4) implies that \( \phi \) is the phase lag relative to the waves in the far field at \( \theta = \pm \pi/2 \) radians.
where $a, b$ are the real and imaginary parts of the coefficient of the exponential term in (6),

$$a(\theta) = \sum_{n=0}^{\infty} \text{Re}(\epsilon_n \cos n\theta),$$

$$b(\theta) = \sum_{n=0}^{\infty} \text{Im}(\epsilon_n \cos n\theta).$$

The relationship that expresses the water-level variation at the shoreline of the paraboloidal island (Fig. 2) is developed with two additional boundary conditions that require continuity in $\zeta$ and in the first derivative of $\zeta$ with respect to $r$ at $r = r_1$. For the solution at the island shore, Homma (1950) has given

$$\zeta(r_0, \theta, t) = \sum_{n=0}^{\infty} 2i^{n+1} \frac{\epsilon_n \alpha_n}{\pi Q_n} \cos n\theta e^{-i\omega t},$$

where

$$\theta = \frac{r_1}{r_0},$$

$$\alpha_n = (1 + n^2 - \tau^2)^{1/2},$$

$$Q_n = -H_n(\tau)(\alpha_n^2 - 1) \sinh(\alpha_n \ln \theta) +$$

$$+ \tau H_n'(\tau)[\alpha_n \cosh(\alpha_n \ln \theta) + \sinh(\alpha_n \ln \theta)]$$

and

$$\tau = \frac{\omega r_1}{(gD_1)^{1/2}}.$$

As in (6), (7), and (8), the real and imaginary parts in the coefficient of the exponential term in (9) may be combined for the amplitude and phase of the diffraction pattern at the shoreline. Note that $\alpha_n$ can be real or imaginary depending upon $n$. For $\alpha_n$ imaginary, the hyperbolic functions in (12) become trigonometric functions.

3. For convenience in computation, the relationship for $Q_n$ is rendered in a somewhat different form from that given by Homma.
The Numerical Analogue. The finite-difference representation of a boundary-value problem is necessarily restricted by the numerical grid system that models the continuum. Optimization of the grid is requisite upon the available storage capacity of the computer system. Within this limitation the primary consideration is an accurate rendition of the waves. In the case of monochromatic plane waves, three grid intervals within one wavelength are required in order to define the wave unambiguously. Polar grid systems having uniform radial and angular increments were chosen for the verification calculations. Since the largest grid spacing in terms of distance occurs at the outer boundary, \( r = r_{\text{max}} \), the condition stated above implies that \( 2\pi r_{\text{max}}/M \) and \( r_{\text{max}}/N \) should be less than one-third of the incident wavelength where \( M \) and \( N \) are the total number of angular and radial increments, respectively. This criterion is used in conjunction with the constraint that the total number of grid points, \( NM \), is fixed at a level compatible with the computer storage. This technique for selecting \( N \) and \( M \) has the effect of compromising the radial-distance requirement for far-field conditions at the outer boundary in order to provide satisfactory angular resolution of the wave system.

The criterion for the numerical stability of the difference equations is evaluated after the wave train and grid are specified. As a stability criterion, the time step between iterations is chosen on the basis of the least-travel time, \( \delta t \), for a long wave passing diagonally through a grid section. The wave train is presumed to move at the Lagrangian wave speed

\[
c = (gDn)^{1/2}
\]

over the distance \( \delta s \) shown in Fig. 3. The travel time should be computed by considering that each polar grid section is an isosceles trapezoid. However, since \( \Delta \theta \) is very small, the following approximation is employed:

\[
\delta t = \frac{n \Delta r}{c} \frac{\Delta \theta}{(n^2 \Delta \theta^2 + 1)^{1/2}}.
\]

The acceptable maximum time step, \( \Delta t \), consistent with stability is taken as a truncated minimum travel time. This method of estimation has been shown to be a sensitive indicator of stability in the case of a polar grid system.

The finite differences employed to approximate the differential operations in (1) are centered in time and space. In this manner the numerical integra-

\[4. \ r = n \Delta r, \ \theta = m \Delta \theta, \ \text{where} \ \Delta r = r_{\text{max}}/N \ \text{and} \ \Delta \theta = 2\pi/M.\]
tion for the value of \( \zeta \) at grid point \((n, m)\) and time level \( k + 1 \) is expressed as:

\[
\zeta_{n,m}^{k+1} = 2 \zeta_{n,m}^k - \zeta_{n,m}^{k-1} + \frac{g}{2} \left( \frac{\Delta t}{\Delta r} \right)^2 \left\{ [D_n - \left( 1 + \frac{1}{n} \right) D_{n+1}] (\zeta_{n+1,m}^k - \zeta_{n,m}^k) - [D_n + \left( 1 - \frac{1}{n} \right) D_{n-1}] (\zeta_{n,m}^k - \zeta_{n-1,m}^k) + \frac{2D_n}{(n\Delta \theta)^2} \left[ (\zeta_{n+1,m}^k - \zeta_{n,m}^k) - (\zeta_{n,m}^k - \zeta_{n-1,m}^k) \right] \right\} .
\]

The computer-storage requirements for the variables \( \zeta \) and \( D \) depend on the maximum number of radial steps, \( N \), and angular steps, \( M \). Since the islands and underwater topography have reflectional symmetry about any line through the origin, calculation with (1a) requires minimum storage of two time steps of the \( \zeta \) field, which are arrays of size \( N(M+1)/2 \) and a permanent-depth array of size \( N \).

The inner boundary analogue is taken as the expression

\[
\zeta_{1,m}^k - \zeta_{2,m}^k = 0,
\]

which effectively places the island boundary midway between \( n = 1 \) and \( n = 2 \). This is accounted for in comparison with the analytic solutions. The working version of the outer-boundary condition should be applied at an \( r_{\text{max}} \) that includes a constant-depth region of considerable extent, \( D = D_e \) for \( r_e \leq r \leq r_{\text{max}} \). This is required in order to fulfill the assumption that the \( \zeta \) field has attained a far-field nature. In the case of the paraboloidal island, this region comprises the outer 77\% of the water area. Returning to eq. (5), the scattered portion of the \( \zeta \) field may be written

\[
\zeta_s = \zeta - \eta = F(\theta) r^{-1/2} e^{ik(r-ct)} .
\]

Alternatively, in the far field, \( \zeta_s \) reduces approximately to the form

\[
\zeta_s(r,\theta,t) = r^{-1/2} G(\theta, r - ct) ,
\]

where \( c = (gD_e)^{1/2} \). The above relationship implies that, for \( r \) sufficiently greater than \( r_e \),

5. \( \zeta_{n,m}^k = \zeta(n\Delta r, m\Delta \theta, k\Delta t) \).
\[ r^{1/2} \zeta_s = \text{constant} \]  

(18)

along the characteristic path

\[ \frac{dr}{dt} = +c, \]  

(19)

with

\[ \frac{d\theta}{dt} = 0. \]  

(20)

The finite difference form of (18) to (20) is a prognostic relationship for \( G \) at \((N, m)\) and time level \( k + 1 \):

\[ r_N^{1/2} (\zeta - \eta)_N^{k+1} = r_{\beta}^{1/2} (\zeta - \eta)_{\beta, m}^k. \]  

(21)

In general, the grid system will not be such that propagation will occur from grid point \( N - 1 \) to \( N \) in the time interval \( \Delta t \). Therefore, the proper value of \( G \) at time level \( k \) must be found at some point, \( \beta \), shown in Fig. 4. With a linear interpolation procedure, the resulting relation for \( \zeta_{N, m}^{k+1} \) is

\[ \zeta_{N, m}^{k+1} = \bar{\zeta}_{N, m}^{k+1} + \left( \frac{N - 1}{N} \right)^{1/2} \frac{c \Delta t}{\Delta r} (\zeta_{N-1, m}^{k} - \bar{\zeta}_{N-1, m}^{k}) + \left( 1 - \frac{c \Delta t}{\Delta r} \right)(\zeta_{N, m}^{k} - \bar{\zeta}_{N, m}^{k}), \]  

(3a)

where \( c \) is the far-field wave speed \((gD_N)^{1/2}\) and \( \bar{\zeta} \) is the real or cosine portion of the incident monochromatic wave, \( \eta \) [eq. (4)].

**Verification.** In order to carry out the verification of the numerical analogue, a number of test calculations were made for both the cylindrical and paraboloidal islands. These tests were designed to investigate how the variations in the parameters representing the islands, waves, and grid coordinates affect the accuracy of the analogue. The verification consisted of a comparison of the amplitude and phase lag of the computed and analytical response at the island shoreline. It should be emphasized that the \( 0^\circ \)-azimuth position on the island.
represents the lee side with respect to the incident waves. The amplitudes of the waves at the shoreline are presented relative to the far-field incident wave, and the reference for the phase-lag calculations is taken as the far-field wave phase at $\theta = 90^\circ$.

The initial computer solutions were for the cylindrical island in water of constant depth. The results of these studies revealed the dependence of the numerical solution on the angular resolution of the wave field. The amplitude and phase-lag graphs (Figs. 5, 6) compare solutions for the island, wave, and grid-system parameters listed in Table I. In Figs. 5 and 6 the open circles represent the computer results and the solid line is drawn from the analytical solution using eqs. (6), (7), and (8). This particular case illustrates the effect
Table I. Cylindrical island, wave and grid systems.

<table>
<thead>
<tr>
<th>ISLAND</th>
<th>Grid System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius ($r_0$)</td>
<td>Radial increments ($N$)</td>
</tr>
<tr>
<td></td>
<td>Radial increment.</td>
</tr>
<tr>
<td></td>
<td>Angular increments ($M$)</td>
</tr>
<tr>
<td></td>
<td>Angular increment.</td>
</tr>
<tr>
<td>Water Depth ($D_0$)</td>
<td>18 km</td>
</tr>
<tr>
<td></td>
<td>4 km</td>
</tr>
<tr>
<td>Wave System</td>
<td>4 min</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Wave System</th>
<th>Grid System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Period ($T$)</td>
<td>Radial increments ($N$)</td>
</tr>
<tr>
<td></td>
<td>Radial increment.</td>
</tr>
<tr>
<td></td>
<td>Angular increments ($M$)</td>
</tr>
<tr>
<td></td>
<td>Angular increment.</td>
</tr>
</tbody>
</table>

of an inadequate angular resolution. Although the error in the relative amplitude is less than $8\%$, the deviation of the phase-lag values from the analytic solution has a maximum of $14\%$ at the azimuth of $45^\circ$. The angular spacing of the grid points at the outer boundary of this system is $23.2$ km while the far-field wavelength is $47.4$ km. Therefore, for a large portion of the grid area the resolution of the wave is considerably below the minimum of three intervals per wave, and the solution suffers accordingly.

The results of three computer calculations for a paraboloidal island are shown in Figs. 7 and 8. For this test the island parameters given in Table II were held constant and the periods of the incident waves were chosen as $4$, $8$, and $12$ minutes. The analytical solutions are generated by eqs. (9), (7), and (8). The computer solutions for amplitude and phase lag are quite close to their analytical counterparts, with the amplitude errors less than $6\%$ and the phase-lag deviations under $2\%$. The degree of successful fit can be seen in the ability of the phase lag deduced from the numerical solution to follow a
change that is almost a discontinuity at an azimuth of 23° for the four-minute wave. It should be noted that the angular resolution has been increased by a factor of 2.25, the angular spacing of grid points at the outer boundary being 5.2 km.

Table II. Paraboloidal island and grid system.

<table>
<thead>
<tr>
<th>ISLAND</th>
<th>GRID SYSTEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shoreline radius ($r_0$)       10 km</td>
<td>Radial increments ($N$)     50</td>
</tr>
<tr>
<td>Shoreline depth ($D_0$)      0.4 km</td>
<td>Radial increment           1 km</td>
</tr>
<tr>
<td>Radius of variable topography ($r_1$) 30 km</td>
<td>Angular increments ($M$) 72</td>
</tr>
<tr>
<td>Depth at $r_1$ ($D_1$)       4 km</td>
<td>Angular increment       5°</td>
</tr>
</tbody>
</table>

There is a consistent tendency for the deviations of the computer solutions to develop at the lee and waveward sides of the island. Moreover, the nature of the deviations appears to be a function of the period of the waves. It is possible that these departures would be reduced for a grid system in which the radial increment varied. However, the overall accuracy of the numerical solutions is sufficiently good so that the analogue of the wave equations and the boundary conditions can be considered successful.

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