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On the Arbitrary Suppression of Vertical Motion in Wind-driven Oceanic Models

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1. Introduction. There has appeared in the oceanographic literature a series of articles in which the response of a three-dimensional ocean to a distribution of surface wind-stress has been investigated under the assumption that the vertical velocity may be entirely neglected in considering the vertical distribution of horizontal flow (hereafter referred to as the \( w = 0 \) assumption). The resulting circulation contains features that are contradictory to the flow in more orthodox models. For example, in discussing the results of one such model, Hidaka (1955: 209) stated, “We do not know an appropriate explanation of the theoretical result that the effect of winds can be felt at a depth several times as large as Ekman’s depth of frictional influence.” The purpose of this note is to demonstrate that such results are spurious, because the model used is not a physically acceptable approximation to a fluid in the range of scales and amplitudes considered. In such cases the \( w = 0 \) model violates the conservation of momentum (i.e. Newton’s Second Law of Motion) in the vertical direction.\(^2\)

In particular, it is well known that large-scale oceanic phenomena are, to a very high degree of accuracy, hydrostatic (Defant 1961: 338; Fofonoff 1962: 331; Proudman 1953: 50; Stommel 1958: 16; Sverdrup et al. 1942: 439). Mathematical flow patterns that violate this powerful restriction are not relevant to the ocean-circulation problem.

The \( w = 0 \) model originated with Goldsborough (1935, hereafter referred to as G.). Goldsborough was aware that the suppression of vertical velocity implied a nonhydrostatic system, but he considered this to be a correct approximation to the wind-driven circulation. Other authors have ignored the vertical equation of motion entirely and have made little or no attempt to justify the model. Such is the case in an article by Hassan (1964, hereafter referred

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2. For variable density models an alternative interpretation in terms of violating the conservation of energy is possible. See p. 173.
to as H.), which appears on pp. 152–167 of this issue. We shall first show that the argument given by Goldsborough to establish his model is fallacious. We shall then show in particular that Hassan's solutions are not compatible with the hydrostatic equation. Goldsborough considers an ocean of uniform density. In Hassan's model the density may be constant or variable.

2. The Goldsborough Model. We omit curvature effects except in the Coriolis terms, and we write in standard notation: the conservation of momentum in the longitudinal, latitudinal, and vertical directions, and the conservation of mass

$$-v u_{xz} - 2\Omega \sin \theta v + 2\Omega \cos \theta w + \frac{I}{\rho_o} p_x = 0,$$

$$-v v_{zz} + 2\Omega \sin \theta u + \frac{I}{\rho_o} p_y = 0,$$

$$-v w_{zz} - 2\Omega \cos \theta u + \frac{I}{\rho_o} p_z = 0,$$

$$u_x + v_y + w_z = 0.$$  

Here \(p\) is the pressure minus the contribution, \(-\rho_o g z\), which is not linked to the field of motion; \(v\) is the vertical eddy viscosity. Following Goldsborough, we have retained the horizontal Coriolis term in (1) and the nonhydrostatic term in (3); these are essentially eqs. (1 A, B, C, D) in G. Goldsborough's approach is to set \(w = 0\) in (1) and (4); these equations are then solved together with (2) for \(u, v, p\). Finally, \(w\) is calculated from (3) and from the known fields, \(u, p\). The argument offered is that this is correct if the amplitude of \(w\) is small, which is claimed to be the case if the total depth of the ocean is small. It is obvious from the outset that the omission of the vertical term in (4) does not depend only on the amplitude of \(w\), since it is the gradient that occurs. Although Goldsborough (1935: 52) performed a calculation of \(w\) from (3), he did not correctly identify the effects of the total depth and the Ekman depth (the parameters \(d\) and the inverse of \(k\) or \(z_e^d\) in G.) on its amplitude (G. 1935: 55, 1). Furthermore, he did not check the ratio of neglected terms in (1) and (4) to typical retained terms. Finally, the argument introduced by Goldsborough for the consideration of an effectively infinitely deep ocean (G. 1935: 55, 3) does not follow logically from his preceding remarks. To clarify these points we shall reconsider the approximation to eqs. (1)–(4) \textit{ab initio}.

We nondimensionalize \(z\) by \(H\), which is to be chosen as the characteristic vertical scale\(^3\) (either the total depth or some naturally occurring length);

\(^3\) Chosen so that the nondimensional independent variables are smooth functions of the scaled vertical coordinate; i.e. the functions and all their derivatives have upper bounds of order unity.
x, y by a single horizontal scale, L; u, v by \( u_0 \); w by \( w_0 \); and \( p \) by \( L q_0 u_0 2\Omega \). The nondimensional forms of \( x, y, z, u, v, w, p \) are written as \( \xi, \eta, \zeta, U, V, W, P \), and eqs. (1)–(4) become

\[
- \left( \frac{D}{H} \right)^2 U_{\xi\xi} - \sin \theta V + \left( \frac{w_0}{u_0} \right) \cos \theta W + P_{\xi} = 0,
\]

\[
- \left( \frac{D}{H} \right)^2 V_{\eta\eta} + \sin \theta U + P_{\eta} = 0,
\]

\[
- \left( \frac{w_0 D^2}{u_0 H L} \right) W_{\zeta\zeta} - \left( \frac{H}{L} \right) \cos \theta U + P_{\zeta} = 0,
\]

\[
U_{\xi} + V_{\eta} + \left( \frac{w_0 L}{u_0 H} \right) W_{\zeta} = 0,
\]

where we have introduced the symbol \( D = \sqrt{v/2\Omega} \), which is essentially the Ekman depth, agreed upon by all authors to be of the order of 100 m. For numbers appropriate to the wind-driven circulation, the correct approximation scheme is due to Ekman and goes as follows. For the surface boundary layer, take \( H = D \), and \( w_0 = (u_0 D)/L \), whence \( (w_0 D^2)/(u_0 H L) = D^2/L^2 \), and (7) becomes hydrostatic within \( o(D/L) \). \( U, V \) are then calculated from (5) and (6), in which the horizontal dependence enters only implicitly (parametrically). In general, the horizontal velocity is not divergence-free, and \( W \) is calculated from (8) in the form \( U_{\xi} + V_{\eta} + W_{\zeta} = 0 \). Next, for the deep geostrophic flow choose \( H \) as the total depth, whence \( D^2/H^2 \ll 1 \), and necessarily \( u_0 = (w_0 L)/H \). The full continuity equation is again retained and (7) remains hydrostatic, but now within \( o(D^2/L^2) \) and \( o(H/L) \). This scheme is self-consistent and the familiar functions that result are indeed smooth.

Goldsborough's approach is to choose \( w_0 \) so that the frictional term in (7) may have a unit coefficient, i.e. \( w_0 = (HLu_0)/D^2 \). He then argues that \( H \) is the total depth and that \( D \) and \( H \) are therefore independent, and that \( w_0 \ll u_0 \) if \( H \) is sufficiently small (G. 1935: 55, 1). Even if this argument were correct and if \( w_0/u_0 \) could be neglected in (5), the term \( (w_0 L)/(u_0 H) = L^2/D^2 \) remains in the continuity equation. On the other hand, Goldsborough then proceeds to identify \( H \) and \( D \) (G. 1935: 55, 3). In this case, if the condition \( w_0 = (HLu_0)/D^2 \) is maintained, we have \( w_0/u_0 = L/D \) and \( (w_0 L)/(u_0 H) = L^2/D^2 \), so that again the wind systems must have a horizontal scale, \( L \), less than the Ekman layer depth, \( D \) (or the flow must occur in a channel 10 m wide).

To illustrate the difference between Goldsborough and Ekman flows in concrete terms, we shall consider the \( w = 0 \) solution to two familiar problems; (i) a uniform surface wind-stress acting upon an ocean with constant Coriolis parameter, and (ii) a wind-stress that varies harmonically acting upon a \( \beta- \)
ocean. Let \( 2 \Omega \sin \theta = f_0 + \beta y \). In the \( u = 0 \) model, \( u_x + v_y = 0 \) and a stream function exists such that \( u = \psi_y, v = -\psi_x \). The fundamental equation, obtained by cross-differentiation of the horizontal momentum equations, is

\[
v(\psi_{xx} + \psi_{yy}) + \beta \psi_x = 0.
\]  

(9)

Note that (9) is only of second order in \( z \) (instead of fourth order), so that only very special boundary conditions can be accepted. This is of course indicative of the incorrect physics. Authors have been careful to apply non-divergent wind-stresses (e.g. Hidaka 1955: 190).

**Example I.** At the sea surface, \( z = 0 \), \( v u_z = 0, v v_z = \tau_0 \); at the sea bottom \( z = -h, u = v = 0 \).

The solution is

\[
u = \tau_0 \frac{v}{v} (z + h), \quad p = \frac{\rho_0 f_0 \tau_0}{v} (z + h) \frac{v}{v},
\]

and \( \psi = [(f_0 \tau_0)/2 v^2] z (z + h) \). This example is illuminating because, even in this case, where \( \psi = 0 \) by Ekman theory, the solution for the horizontal velocities of the "\( \psi = 0 \) model" differs drastically from the Ekman flow; only one flow component occurs, and this penetrates to the sea bottom. The reader can easily convince himself of the scaling inconsistency by evaluating neglected terms in (1) and (4).

**Example II.** At \( z = 0 \), \( v \psi_{zx} = -\tau_0 e^{i(nx + my)}; \psi \to 0 \) as \( z \to \infty \).

Here the solution is \( \psi = i(\tau_0/v \pi n) e^{i(nx + my + ax + my)} \), where \( \alpha^2 = i(\beta n/[v(n^2 + m^2)]) \) and the decaying square root of \( i \) must be chosen. The scaling inconsistency is of the type discussed above when \( H = D \). A point of interest is that, unlike the Ekman solution, the depth of penetration of the flow does depend on the horizontal scales and becomes deeper as these decrease. If a simple harmonic wind system has a horizontal scale comparable to \( f_0/\beta \), then for a given \( \nu \) the depth of penetration will be essentially comparable to the Ekman depth (Goldsmidhborough's result). However, when meridional boundaries are introduced, the formation of a western boundary current produces a narrow longitudinal phenomenon that in turn forces deeper flow (Hidaka 1955, Hassan 1958). It is instructive here also to compute the pressure field.

3. **The Hassan Model.** Following Hassan 1964, we define a dynamic eddy viscosity, \( A = \nu v \), and we introduce a density mean and anomaly by writing \( \rho = \rho_0 (1 + s) \), to distinguish where it is correct to treat the density as variable or constant (this distinction is made tacitly in H., vide his eq. H-6 ff). Retaining the vertical velocity and the third equation of motion, but otherwise following Hassan's assumptions, we have the equations
Eq. (14) is derivable from the conservation of heat and salt, under the assumption of negligible turbulent mixing. We shall refer to it simply as the heat equation. These equations express the basic physics of the model, and any further assumption must be compatible with the full set. An example of a compatible assumption is a homogeneous density model in which \( s = 0 \), (14) is identically satisfied, and (10)–(13) determine the four remaining fields.

The approach of Hassan has been to regard (10), (11) and (13), (14) with \( w = 0 \),

\[
\begin{align*}
&u_x + v_y = 0, \\
&us_x + vs_y + ws_z = 0
\end{align*}
\]

as a set of four equations for \( u, v, p, s \). He ignores (12) entirely, although this is a \( p, s \) relationship. Thus there are in actuality more equations than unknowns. In fact, Hassan ignores the heat equation and merely calculates \( u \) and \( v \) from (9) with \( A = v_0 \). We shall show that his solutions are not compatible with both (12) and (16), and thus are not approximate solutions to the set (10)–(14).

According to (9), a separation of the \( y \)-dependence in the form

\[
\psi = \sin \frac{\pi y}{b} \varphi(x,z), \quad u = \cos \frac{\pi y}{b} \cos \frac{\pi x}{b} \bar{u}(x,z), \quad v = -\sin \frac{\pi y}{b} \varphi_x x = \sin \frac{\pi y}{b} \bar{v}(x,z)
\]

is allowed. Hassan’s solution (following his eq. H-6) has this form. To consider the compatibility of these solutions with eqs. (12) and (16), we distinguish two cases: constant and variable density.

**Case I, \( s = 0 \).** Eq. (16) is satisfied identically and (12) implies \( p_{xz} = p_{yz} = 0 \); whence, from (10) and (11),
Substituting from (17) into (18), multiplying by \( \sin \pi y/b \), and integrating from 0 to \( b \), we find \( \tilde{v}_x = 0 \). Similarly from (19), \( \tilde{u}_z = 0 \). Hassan’s solutions clearly violate these conditions, which are necessary for compatibility with the heat and hydrostatic equations.

CASE II, \( s \neq 0 \). We obtain \( s_x = -(1/\rho_0) p_{xx} \), \( s_y = -(1/\rho_0) p_{yy} \) from (10), (11), (12); substituting into (16), we obtain

\[
\frac{u (fv_x + Au_{xx}) - v (fu_x - Av_{xx})}{\rho_0} = 0.
\]

By (17), this may be written as

\[
\frac{2 \pi y}{b} f (\tilde{u}_x - \tilde{v}_x) + A \frac{2}{b} (\tilde{u}_{xx} + \tilde{v}_{xx}) + \cos \frac{2 \pi y A}{b} \left( \tilde{u}_{zxx} - \tilde{v}_{zxx} \right) = 0.
\]

Multiplying by \( \sin \left((2 \pi y)/b\right) \) and integrating from 0 to \( b \), we find \( \tilde{u}_x - \tilde{v}_x = 0 \) or \( \tilde{u}(x,z) = k(x) \tilde{v}(x,z) \). Hassan’s solutions do not obey this restriction.

The demonstration of incompatibility via a discussion of the \( y \)-separation of the solutions has no particular significance; it has merely been used for simplicity. The \( y \)-separation used by Hassan implies that there is no balance between the stress and Coriolis terms in the momentum equation; i.e. each term is supported separately by a contribution from the pressure. In an Ekman layer the stress and Coriolis terms are in exact balance, and represent the only correct approximation.

4. Discussion. The fact that the \( w = 0 \) solutions are not in hydrostatic balance if the heat equation is satisfied means that the only way such motion could be realized would be to have an arbitrary distribution of body force in the vertical direction (momentum sources and sinks) so as to just “patch up” the momentum balance. Alternatively, the hydrostatic balance could be maintained and eq. (16) altered to include an arbitrary distribution of heat sources and sinks so as to just “patch up” the heat balance. Both situations are physically unacceptable.

Finally it should be mentioned that oceanic models in which the vertical velocity is not explicitly discussed are sometimes mentioned for the support of the \( w = 0 \) models (Sverdrup 1947, Munk 1950). In these models, vertical velocity occurs implicitly; under physically acceptable assumptions it is deduced that the horizontal transport field does not diverge. Stommel (1948) explicitly uses the hydrostatic relationship.
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