The *Journal of Marine Research* is an online peer-reviewed journal that publishes original research on a broad array of topics in physical, biological, and chemical oceanography. In publication since 1937, it is one of the oldest journals in American marine science and occupies a unique niche within the ocean sciences, with a rich tradition and distinguished history as part of the Sears Foundation for Marine Research at Yale University.

Past and current issues are available at journalofmarineresearch.org.
The General Oceanic Circulation;
Some Computations, a Working Hypothesis,
and Proposed Tests

El Sayed Mohamed Hassan

Geophysical Sciences Laboratory
Department of Meteorology and Oceanography
New York University

ABSTRACT

An analytic solution for the three-dimensional structure of horizontal velocity has been computed for a parallelepipedic model ocean with certain restrictions. Observed features that repeated themselves when the model parameters were changed have been generalized, and it is hypothesized that some of the persistent features obtained from the computations exist in the real ocean.

1. Introduction. Although interest in the structure of three-dimensional currents is not new (Ekman 1905, Nomizu 1933), the impossibility of solving the full hydrodynamic equations analytically and the difficulty of testing, by direct oceanographic measurements, the solutions derived for simplified current systems have so dampened the enthusiasm of oceanographers that they have not pursued this line of study to any great extent. Instead of the three-dimensional solution, the vertically integrated transport has received much attention, because (i) it avoids direct velocity measurements in the ocean, and (ii) the equations are more tractable. Knowledge of the transport, however,

1. This work has been sponsored by the U.S. Navy Hydrographic Office under Contract N-62306-794. Contribution No. 1 of the Geophysical Sciences Laboratory, Department of Meteorology and Oceanography, New York University, New York 53, N.Y.
2. On leave of absence from Cairo University.

EDITOR'S NOTE

When Dr. Hassan's original manuscript was reviewed for publication in the Journal of Marine Research, there were pronounced differences of opinion among at least some oceanographers regarding certain fundamental concepts employed in his paper. I therefore proposed to Dr. Hassan that we publish concurrently with his paper another article by one of the reviewers whose views were contrary to his. Dr. Hassan generously agreed, and the following paper was prepared by Dr. Allan Robinson. I wish to express my appreciation to Dr. Hassan for his cooperation and particularly for his patience with the inevitable delay incurred in publication, and to Dr. Allan Robinson for his generous cooperation in preparing the contribution on pages 168-174. - Yngve H. Olsen.
cannot be substituted for knowledge of the detailed velocity structure. The increasing need for this knowledge along with two technical developments have helped to renew the efforts directed toward detailed studies of velocity structure. These developments are the advent of the electronic computer and the advances made in the direct measurement of deep currents.

In a previous attempt to determine the structure of the horizontal currents in three dimensions (Hassan 1958), some interesting patterns appeared. However, it was not apparent how and to what extent the patterns would be dependent upon the numerical values of the parameters used; and the variability of the patterns was not obvious from consideration of the analytical solution obtained. Once more computation of the solution was undertaken, this time with different combinations of the parameters, and the results point to the hypothesis discussed on pp. 157–160.

2. Fundamental Equations, Assumptions, Conditions, and Derivations. The fundamental equations are:

\[-f \varrho \nu = - \frac{\partial \varrho}{\partial x} + A \frac{\partial^2 u}{\partial z^2},\]

\[f \varrho u = - \frac{\partial \varrho}{\partial y} + A \frac{\partial^2 v}{\partial z^2},\]

\[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\]

\[\text{and } u \frac{\partial \varrho}{\partial x} + v \frac{\partial \varrho}{\partial y} = 0,\]

where \(x, y,\) and \(z\) are the Cartesian coordinates increasing eastward, northward, and downward, respectively, \(u\) and \(v\) the velocities in the \(x-\) and \(y-\) directions, \(f\) the Coriolis parameter, \(\varrho\) the density of the fluid, \(A\) the eddy viscosity coefficient, and \(p\) the pressure.

The assumptions implied in the equations are: (i) steady motion, i.e. \(\partial/\partial t = 0;\) (ii) no vertical velocity, i.e. \(w = 0;\) (iii) negligible field accelerations compared with the terms retained in the equations, i.e. \(u(\partial u/\partial x), u(\partial v/\partial x), v(\partial u/\partial y), v(\partial v/\partial y) \ll (f \varrho \nu, f \varrho u);\) (iv) nondivergent motion; (v) incompressible fluid; (vi) no heat transfer to or from the system; and (vii) only the vertical eddy viscosity is taken into account and its coefficient is considered constant.

The boundary conditions are as follows: (i) Boundaries are streamlines, i.e. \(u = 0\) at \(x = 0, l;\) \(v = 0\) at \(y = 0, b;\) for a rectangular ocean between \(x = 0, l\) and \(y = 0, b.\) (ii) Velocities vanish at the bottom, i.e. \(u = v = 0\) at \(z = h.\)
(iii) A zonal wind stress acts at the surface and varies only with latitude like a cosine function, i.e. \( \tau_x = -2 \cos (\pi y/b) \) and \( \tau_y = 0 \). (iv) The stress at the surface is equal to the wind stress, i.e.

\[
A(\partial u/\partial z)_{z=0} = \tau_x = \tau_0 \cos \pi y/b = -2 \cos \pi y/b \text{ dynes/cm}^2, \quad \text{and} \quad A(\partial v/\partial z)_{z=0} = 0.
\]

The solution assumes that:

\[
u = \sum v_i \cos \left( \frac{(2i-1)\pi z}{2h} \right), \quad \text{where} \quad v_i = \frac{2}{h} \int_{0}^{h} \nu \cos \left( \frac{(2i-1)\pi z}{2h} \right) dz;
\]

\[
\frac{\partial^2 u}{\partial z^2} = \sum \mu_i \cos \left( \frac{(2i-1)\pi z}{2h} \right), \quad \text{where} \quad \mu_i = \frac{2}{h} \int_{0}^{h} \frac{\partial^2 u}{\partial z^2} \cos \left( \frac{(2i-1)\pi z}{2h} \right) dz;
\]

\[
\frac{\partial^2 v}{\partial z^2} = \sum v_i \cos \left( \frac{(2i-1)\pi z}{2h} \right), \quad \text{where} \quad v_i = \frac{2}{h} \int_{0}^{h} \frac{\partial^2 v}{\partial z^2} \cos \left( \frac{(2i-1)\pi z}{2h} \right) dz.
\]

It can be shown by using the boundary conditions at the surface and at the bottom that:

\[
\mu_i = \left\{ \frac{2\tau_x}{Ah} - \frac{(2i-1)^2 \pi^2}{4h^2} u_i \right\},
\]

\[
v_i = \left\{ -\frac{(2i-1)^2 \pi^2}{4h^2} v_i \right\}.
\]

The vorticity equation developed from (1) and (2), utilizing (3) and (4), is thus

\[
\left( \sum_i \phi \beta \frac{\partial \psi_i}{\partial x} - \frac{2}{h} \frac{\partial \tau_x}{\partial y} + \frac{(2i-1)^2 \pi^2}{4h^2} A \left[ \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} \right] \right) \cos \left( \frac{(2i-1)\pi z}{2h} \right) = 0, \quad (5)
\]

where \( \psi_i \) is a stream function defined by \( \partial \psi_i/\partial y = u_i \), \( \partial \psi_i/\partial x = -v_i \); and where \( \beta = \partial f/\partial y \).

As (5) is valid for every depth, the coefficient of every “\( \cos [(2i-1)\pi z]/2h \)” should vanish identically. We thus get the system

\[
\frac{\partial \psi_i}{\partial x} = \frac{2}{h} \frac{\partial \tau_x}{\partial y} - \frac{(2i-1)^2 \pi^2}{4h^2} A \left( \frac{\partial^2 \psi_i}{\partial x^2} + \frac{\partial^2 \psi_i}{\partial y^2} \right). \quad (6)
\]
The solution of (6) with the specified boundary conditions is:

$$\psi_i = \gamma_i \left( \frac{b}{\pi} \right)^2 \sin \frac{\pi y}{b} \left[ q_i e^{G_i x} + s_i e^{D_i x} - 1 \right],$$

where

$$\gamma_i = \frac{8 \tau_0 h}{A b \pi (2i - 1)^2} = \frac{16 h}{A b \pi (2i - 1)^2},$$

$$C_i = \frac{-\delta_i}{2} + \sqrt{\frac{(\delta_i)^2}{4} + \frac{\pi^2}{b^2}},$$

$$D_i = \frac{-\delta_i}{2} - \sqrt{\frac{(\delta_i)^2}{4} + \frac{\pi^2}{b^2}},$$

$$\delta_i = \frac{2 \beta h^2}{A \pi^2 (2i - 1)^2},$$

$$q_i = \frac{1 - e^{D_i l}}{e^{C_i l} - e^{D_i l}}, \text{ and } s_i = 1 - q_i.$$

The expressions for \( \psi, u, \) and \( v \) then become:

$$\psi = \left( \frac{b}{\pi} \right)^2 \sin \frac{\pi y}{b} \cdot \sum_i \gamma_i \left[ q_i e^{G_i x} + s_i e^{D_i x} - 1 \right] \cos \frac{(2i - 1)\pi z}{2h},$$

$$u = \left( \frac{b}{\pi} \right) \cos \frac{\pi y}{b} \cdot \sum_i \gamma_i \left[ q_i e^{G_i x} + s_i e^{D_i x} - 1 \right] \cos \frac{(2i - 1)\pi z}{2h},$$

$$v = \left( \frac{b}{\pi} \right)^2 \sin \frac{\pi y}{b} \cdot \sum_i \gamma_i \left[ G_i q_i e^{G_i x} + D_i s_i e^{D_i x} \right] \cos \frac{(2i - 1)\pi z}{2h}.$$

The transport stream function and the transport components in the \( x- \) and \( y- \)directions are expressed by:

$$\Psi = \int_P \psi \, dz = 2h \left( \frac{b}{\pi} \right)^2 \sin \frac{\pi y}{b} \cdot \sum_i \gamma_i \left[ q e^{G_i x} + s_i e^{D_i x} - 1 \right] \frac{(-1)^{i-1}}{(2i - 1)},$$

$$U = \int_P u \, dz = 2h \left( \frac{b}{\pi} \right) \cos \frac{\pi y}{b} \cdot \sum_i \gamma_i \left[ q_i e^{G_i x} + s_i e^{D_i x} - 1 \right] \frac{(-1)^{i-1}}{(2i - 1)},$$

$$V = \int_P v \, dz = 2h \left( \frac{b}{\pi} \right)^2 \sin \frac{\pi y}{b} \cdot \sum_i \gamma_i \left[ C_i q_i e^{G_i x} + D_i s_i e^{D_i x} \right] \frac{(-1)^{i-1}}{(2i - 1)}.$$
and

\[ V = \int_{0}^{\pi} \frac{2h}{b} \left( \frac{b}{\pi} \right)^{2} \sin \frac{\pi y}{b} \sum_{i} \psi \left [ C_{i} q_{i} e^{C_{i}x} + D_{i} s_{i} e^{D_{i}x} \right ] \left( \frac{-1}{2i-1} \right). \]

3. Computation. Ten oceanic models with various combinations of width, depth, and exchange coefficients (eddy viscosity coefficients) were considered. The first 50 terms in the Fourier expansion were calculated for \( \psi, u, v, \Psi, U, \) and \( V; \) however, after the computation was completed, it became apparent that using 50 terms was unnecessary. Actually, in some cases, the last terms were incorrect due to the precision limit of the computer memory (single precision was used throughout). This did not affect the transport stream functions, because, in the Fourier series, the \( n \)th term was divided by \( (2n - 1) \) as compared with the similar term in the velocity stream-function series. Therefore, valid remarks can be made regarding the transport functions for all ten oceanic models, but only five of these yield useful results for the velocities. Those models that gave useful results for the velocities are compiled first in Table I, which also contains the dimensions of different oceans considered, the value for the exchange coefficients, and some characteristics of the solution.

**TABLE I. Parameters Used in the Computation, and Some Results.**

<table>
<thead>
<tr>
<th>Model</th>
<th>Breadth (km)</th>
<th>Width (km)</th>
<th>Depth (m)</th>
<th>Exchange Coefficient (cm(^{-4}) sec(^{-1}) g)</th>
<th>Max. Transport Stream Function (10^{6} \text{ m}^{3}/\text{sec})</th>
<th>Value of Max. Stream Function (10^{2} \text{ m}^{2}/\text{sec})</th>
</tr>
</thead>
<tbody>
<tr>
<td>I*...</td>
<td>3142 ( \sim \pi \times 1000 )</td>
<td>5,000</td>
<td>500</td>
<td>50</td>
<td>64.7</td>
<td>136.0</td>
</tr>
<tr>
<td>II*...</td>
<td>3142</td>
<td>5,000</td>
<td>500</td>
<td>100</td>
<td>68.9</td>
<td>96.9</td>
</tr>
<tr>
<td>III*...</td>
<td>3142</td>
<td>5,000</td>
<td>500</td>
<td>500</td>
<td>68.1</td>
<td>44.0</td>
</tr>
<tr>
<td>IV*...</td>
<td>3142</td>
<td>500</td>
<td>500</td>
<td>100</td>
<td>6.96</td>
<td>5.9</td>
</tr>
<tr>
<td>V*...</td>
<td>3142</td>
<td>50,000</td>
<td>500</td>
<td>100</td>
<td>625.</td>
<td>391.0</td>
</tr>
<tr>
<td>VI...</td>
<td>3142</td>
<td>5,000</td>
<td>5000</td>
<td>50</td>
<td>64.0</td>
<td>91.7</td>
</tr>
<tr>
<td>VII...</td>
<td>3142</td>
<td>5,000</td>
<td>5000</td>
<td>100</td>
<td>63.4</td>
<td>74.8</td>
</tr>
<tr>
<td>VIII...</td>
<td>3142</td>
<td>5,000</td>
<td>5000</td>
<td>500</td>
<td>64.1</td>
<td>39.5</td>
</tr>
<tr>
<td>IX...</td>
<td>3142</td>
<td>500</td>
<td>5000</td>
<td>100</td>
<td>6.42</td>
<td>5.0</td>
</tr>
<tr>
<td>X...</td>
<td>3142</td>
<td>50,000</td>
<td>5000</td>
<td>100</td>
<td>641.</td>
<td>360.0</td>
</tr>
</tbody>
</table>

* Valid models; see explanation in text under Computation.

Note that all five models not affected by the rate of convergence of Fourier series are the shallow ones (500 m). This suggests that the convergence rate of the Fourier series diminishes as the depth of the ocean increases, indicating that the vertical scale of current structure is not proportional to depth.

The computations were carried out for 100 equally spaced depth levels between the surface and the bottom. At each level, the solution was determined
as shown by the grid in Fig. 1. Model 1 serves as an example to demonstrate the different features that resulted (see Figs. 2–14).

4. Results. The effects of variable parameters on the features of the circulation are summarized in Table II.

5. Discussion. The distribution of the vertically integrated transport resembles the pattern for surface circulation, and this masked large current variations with depth. Under each surface current, except at the extreme western boundary, the direction of water movement reversed at least once. Sometimes this reversal was accomplished by passage through zero (Fig. 13), but in general it resulted from a continuous rotation of the current vector (Fig. 14). Possibly this points to an important rule: in the presence of closed boundaries, the Ekman frictional layer extends beyond the traditional "depth of frictional influence," possibly down to the bottom. To wit, the Ekman frictional depth appropriate to Model 1 is about 21 m (assuming no boundaries and a homogeneous ocean). In Figs. 13 and 14 it can be seen that the depth where the velocity reaches 5% of its surface value is much greater than 21 m. Should this rule prove to be valid, then the present concepts of the ocean as a layer of frictional influence overlying a

3. G. Neumann, in a private discussion, has suggested that the complicated pattern with depth may result from restriction of vertical motion.
**TABLE II. Effects of Changing Parameters.**

<table>
<thead>
<tr>
<th>Feature</th>
<th>Depth</th>
<th>Width</th>
<th>Exchange Coefficient</th>
<th>Wind-Stress Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertically integrated transport</td>
<td>Almost no change*</td>
<td>Almost proportional</td>
<td>Almost no change*</td>
<td>Exactly proportional</td>
</tr>
<tr>
<td>Relative position of maximum transport</td>
<td>Shifts westward with increasing depth</td>
<td>Shifts westward with increasing width</td>
<td>Shifts westward with decreasing exchange coefficient</td>
<td>No effect</td>
</tr>
<tr>
<td>stream function</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Relative position of maximum surface</td>
<td>Shifts westward with increasing depth</td>
<td>Shifts westward with increasing width</td>
<td>Shifts westward with decreasing exchange coefficient</td>
<td>No effect</td>
</tr>
<tr>
<td>stream function</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Magnitude of maximum surface stream</td>
<td>Decreases with increasing depth</td>
<td>Increases with increasing width</td>
<td>Decreases with increasing exchange coefficient</td>
<td>Exactly proportional</td>
</tr>
<tr>
<td>function</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maximum value of surface velocity</td>
<td>Almost proportional</td>
<td>Almost proportional</td>
<td>Decreases with increasing exchange coefficient</td>
<td>Exactly proportional</td>
</tr>
<tr>
<td>(meridional velocity at western boundary)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* A change in oceanic depth or exchange coefficient by a factor of 10 produced a change in the vertically integrated transport of less than 5 per cent.
much thicker layer of geostrophic flow and supplying it with sources and sinks would become rather artificial.

These results might also contribute to an understanding of the complex structure of the oceanic circulation that is just now emerging from the relatively few direct current measurements (Swallow and Hamon 1959). It should be pointed out that these measurements are not always reproducible, indicating a great variation in the currents either with time or space; the estimated uncertainty in depth is about 100 m (Swallow and Hamon). This could very well place the neutral buoys in two different regimes of water movement.

6. Working Hypothesis and Tests. For a working hypothesis it is assumed that some of the features exhibited in the solutions obtained (Table u) are inherent in the current structure of the oceans. These features are: (i) Currents in the ocean flow in closed cells. (ii) For horizontal cross sections in any one cell, a line that passes through similar points of different cross sections will, whenever possible, incline westward with depth. (iii) Immediately to the east of each cell, either the eastern boundary or another cell with the opposite sense of rotation will be found.

From an examination of these features, the following corollaries can be derived: (i) A countercurrent will be encountered under every current. In actual observations the countercurrent will sometimes be masked because of its weakness or because of ever-present transients.

(ii) The core of any zonal current will rise toward the surface in an eastward direction.

(iii) The spiral that results from plotting the end of the velocity vector in any vertical line will have an opposite sense of rotation in the northern half of any given cell compared with the southern half.

The second and third corollaries offer suitable means to test the hypothesis. By measuring the steady currents along any vertical in the northern and southern half of a well-defined gyre, one could easily ascertain whether the two spirals display an opposite sense of rotation. Current meters suspended from anchored buoys would be splendid for this test. Another test, though more elaborate, could be performed by measuring the depth of a well-defined undercurrent at a series of zonally distributed stations to ascertain whether it has the required inclination in the east-west direction. This also could be done with a string of current meters suspended from a series of anchored buoys, if they were oriented along an east-west line.

7. Support. Some support for the hypothesis is found in both theory and observations. The work of Goldsborough (1934) indicates that, on a water-covered rotating globe, closed wind systems produce closed current systems that are displaced westward. If one considers that the surface current system is closed by boundary limitations, and if the system acts on underlying layers
in the same way as the wind, as described in Goldsborough's model, the rest of the structure then follows.

Although the hypothesis predicts the existence of countercurrents, two major examples of which have been observed (Swallow and Worthington 1957, Montgomery and Stroup 1962), it does not explain all other characteristics.

Further, the characteristics explained by the hypothesis can have other explanations. For example, (i) The countercurrent under the Gulf Stream might result from thermohaline causes (Stommel 1957). (ii) The rise in the core of the equatorial undercurrent might be attributed to (a) the nonlinear terms in the equations of motion coupled with a decrease eastward in the exchange coefficients (Charney 1960: fig. 1), (b) a variation in wind stress, or (c) the temperature gradient across the equator (Veronis 1960).

8. Acknowledgments. The author wishes to express his appreciation to Dr. David M. Garner, with whom a running discussion was going on throughout all phases of the work reported here, and to Dr. G. Neumann and Dr. W. J. Pierson, Jr., for helpful comments.

REFERENCES

Charney, J. G.  

Ekman, V. W.  

Goldsborough, G. R.  

Hassan, E. S. M.  

Knauss, J. A.  

Montgomery, R. B., and E. D. Stroup  

Nomizu, T.  

Stommel, Henry  

Swallow, J. D., and B. V. Hamon  

Swallow, J. D., and L. B. Worthington  

Veronis, George  
Figure 2. Transport streamlines in Ocean I. Units are $10^6 \text{ m}^3/\text{sec}$.

Figure 3. Streamlines at the surface of Ocean I. Units are $10^3 \text{ m}^2/\text{sec}$. 
Figure 4. Streamlines at the depth of 50 m in Ocean 1. Units are $10^3 \text{ m}^2/\text{sec}$.

Figure 5. Streamlines at the depth of 100 m in Ocean 1. Units are $10^3 \text{ m}^2/\text{sec}$. 
Figure 6. Streamlines at the depth of 150 m in Ocean 1. Units are $10^3 \text{m}^3/\text{sec}$.

Figure 7. Streamlines at the depth of 200 m in Ocean 1. Units are $10^3 \text{m}^3/\text{sec}$.
Figure 8. Streamlines at the depth of 250 m in Ocean 1. Units are \(10^3 \text{ m}^3/\text{sec}\).

Figure 9. Streamlines at the depth of 300 m in Ocean 1. Units are \(10^3 \text{ m}^3/\text{sec}\).
Figure 10. Streamlines at the depth of 350 m in Ocean 1. Units are $10^3 \text{m}^3/\text{sec}$.

Figure 11. Streamlines at the depth of 400 m in Ocean 1. Units are $10^3 \text{m}^3/\text{sec}$.
Figure 12. Streamlines at the depth of 450 m in Ocean 1. Units are $10^3 \text{ m}^2/\text{sec}$.

Figure 13. Variations of meridional velocity with depth at selected points in Ocean 1. Units are cm/sec.
Figure 14. Hodograph of the velocity vector with depth at selected points in Ocean 1. Units are cm/sec.