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BOUNDARIES IN THE THEORY OF OCEAN CURRENTS

By

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ABSTRACT

Described in this paper is a technique of using artificial water boundaries in theories of ocean currents. As boundary values, geostrophic currents are suggested. This technique may hold promise for other applications in theoretical discussions of ocean currents.

1. Boundaries in the Ocean. In order to simplify mathematical analysis, theoretical discussions of ocean currents, excepting that of Munk and Carrier (1950), have been confined to rectangular or quadrilateral models. Serious difficulties arise when attempts are made to expand these simplified conditions to actual oceans with irregular boundaries.

Except near the Equator, geostrophic currents can be calculated from the observed distribution of mass. But geostrophic currents, as practical approximations to actual currents, are inadequate if we must consider both vertical and horizontal friction, and we are faced with complicated boundary value problems.

At the surface, velocity components are determined by observation, or it is assumed that wind traction balances surface stress.

At solid coastal boundaries, velocities perpendicular to such boundaries must vanish, and, in the presence of lateral eddy viscosity, motion parallel to them must vanish also. These boundary conditions can be used for any part of an ocean bounded by coasts.

But we still have to propound boundary conditions where the boundary is water, and this paper attempts to show that we can probably compute the horizontal flow pattern in a water-bound rectangular or quadrilateral ocean.

2. Geostrophic Boundary Conditions. In a previous paper (Hidaka and Nagata, 1958), it was shown that, in a section without coastal boundaries, the geostrophic currents closely approximate the actual currents and that these geostrophic current velocities provide us
with convenient boundary conditions. Since the geostrophic flow only approximates the true circulation, it is desirable to derive differential equations in which boundary conditions are of minimal importance.

3. Theory. Taking the x-axis eastward and the y-axis northward, and assuming a given pressure distribution in a sea surface swept by prevailing winds, the hydrodynamic equations of stationary ocean currents are expressed by

$$
\begin{align*}
A_1 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) &+ \mu \frac{\partial^2 u}{\partial z^2} + 2 \omega \sin \varphi \varphi v = \frac{\partial p}{\partial x}; \\
A_1 \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) &+ \mu \frac{\partial^2 v}{\partial z^2} - 2 \omega \sin \varphi \varphi u = \frac{\partial p}{\partial y};
\end{align*}
$$

(1)

where \( u \) and \( v \) are components of current velocity parallel to the \( x \)-axis and \( y \)-axis, \( p \) is the pressure, \( \mu \) and \( A_1 \) are the coefficients of vertical and lateral eddy viscosity, \( \varphi \) is the density of sea water, \( \omega \) is the angular velocity of the earth, and \( \varphi \) is the geographic latitude.

Now let \( u \) and \( v \) represent components of a current relative to a certain reference level \( z = h \). Then both \( \frac{\partial p}{\partial x} \) and \( \frac{\partial p}{\partial y} \) are referred to the same reference level.

If we multiply the second part of (1) by \( i = \sqrt{-1} \) and add the product to the first part,

$$
A_1 \left( \frac{\partial^2 \bar{W}}{\partial x^2} + \frac{\partial^2 \bar{W}}{\partial y^2} \right) + \mu \frac{\partial^2 \bar{W}}{\partial z^2} - i 2 \omega \sin \varphi \varphi \bar{W} = \frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y},
$$

(2)

where

$$
\bar{W} = u + iv.
$$

(3)

On the surface \( (z = 0) \) we have

$$
z = 0; \quad -\mu \frac{\partial \bar{W}}{\partial z} = \tau_x + i \tau_y,
$$

(4)

and at the reference level \( (z = h) \) we have

$$
z = h: \quad \bar{W} = 0.
$$

(5)
On the water boundaries, \( x = 0, a \) and \( y = 0, b \), geostrophic currents are computed from mass distribution. In mathematical language, we have

\[
\begin{align*}
(1) \text{ at the boundary } x &= 0 \\
\overline{W} &= G(0, y, z); \\
(2) \text{ at the boundary } x &= a \\
\overline{W} &= G(a, y, z); \\
(3) \text{ at the boundary } y &= 0 \\
\overline{W} &= G(x, 0, z); \\
(4) \text{ at the boundary } y &= b \\
\overline{W} &= G(x, b, z),
\end{align*}
\]

where \( G(x, y, z) \) represents the geostrophic current at \( x, y, \) and \( z \).

Now, if \( D \) represents the geopotential distance over the reference surface,

\[
\frac{\partial p}{\partial x} + i \frac{\partial p}{\partial y} = \varrho \left( \frac{\partial D}{\partial x} + i \frac{\partial D}{\partial y} \right),
\]

and the distribution of \( D \) gives the so-called relative dynamic topography. Then

\[
\frac{A_l}{\varrho} \left( \frac{\partial^2 \overline{W}}{\partial x^2} + \frac{\partial^2 \overline{W}}{\partial y^2} \right) + \frac{\mu}{\varrho} \frac{\partial^2 \overline{W}}{\partial z^2} - i 2 \omega \sin \varphi \overline{W} = \frac{\partial D}{\partial x} + i \frac{\partial D}{\partial y}
\]

with the boundary conditions (4), (5) and (6).

Now it is possible to expand \( \overline{W} \) and \( D \) in a series of forms:

\[
\overline{W}(x, y, z) = \sum_n \overline{W}_n(x, y) \cos \left( \frac{(2n-1)\pi z}{2h} \right),
\]

where

\[
\overline{W}_n(x, y) = \frac{2}{h} \int_0^h \overline{W}(x, y, \zeta) \cos \left( \frac{(2n-1)\pi \zeta}{2h} \right) \, d\zeta;
\]

and

\[
D(x, y, z) = \sum_n D_n(x, y) \cos \left( \frac{(2n-1)\pi z}{2h} \right),
\]

where

\[
D_n(x, y) = \frac{2}{h} \int_0^h D(x, y, \zeta) \cos \left( \frac{(2n-1)\pi \zeta}{2h} \right) \, d\zeta.
\]
Further assume
\[ \frac{\mu \partial^2 \tilde{W}}{\rho \partial z^2} = \sum_n \cos \left( \frac{(2n-1)\pi z}{2h} \right) \frac{2}{h} \int_0^h \frac{\mu \partial^2 \tilde{W}}{\rho \partial \zeta^2} \cos \left( \frac{(2n-1)\pi \zeta}{2h} \right) d\zeta; \]
then, by Stokes method,
\[ \frac{2}{h} \int_0^h \frac{\mu \partial^2 \tilde{W}}{\rho \partial \zeta^2} \cos \left( \frac{(2n-1)\pi \zeta}{2h} \right) d\zeta = \frac{2}{h} (\tau_x + i\tau_y) - \frac{(2n-1)^2 \pi^2 \mu}{4h^2 \rho} \tilde{W}_n(x, y); \]
therefore
\[ \frac{\mu}{\rho} \frac{\partial^2 \tilde{W}}{\partial z^2} = \sum_n \left\{ \frac{2}{h} (\tau_x + i\tau_y) - \frac{(2n-1)^2 \pi^2 \mu}{4h^2 \rho} \tilde{W}_n(x, y) \right\} \cos \left( \frac{(2n-1)\pi z}{2h} \right). \quad (10) \]

Substituting (8), (9) and (10) in (7), we have an equation for \( \tilde{W}_n(x, y) \) as follows:
\[
\frac{A_1}{\rho} \left( \frac{\partial^2 \tilde{W}_n}{\partial x^2} + \frac{\partial^2 \tilde{W}_n}{\partial y^2} \right) - \left\{ \frac{(2n-1)^2 \pi^2 \mu}{4h^2 \rho} + i \cdot 2\omega \sin \varphi \right\} \tilde{W}_n
\]
\[
= \frac{\partial D_n}{\partial x} + i \frac{\partial D_n}{\partial y} - \frac{2}{h \rho} (\tau_x + i\tau_y).
\]
\[
(n = 1, 2, 3, \ldots)
\]

Now we have a close relationship
\[ d^2 \left( \frac{\partial^2 \tilde{W}_n}{\partial x^2} + \frac{\partial^2 \tilde{W}_n}{\partial y^2} \right) = \tilde{W}_n(x + d, y) + \tilde{W}_n(x - d, y) + \tilde{W}_n(x, y + d) + \tilde{W}_n(x, y - d) - 4 \tilde{W}_n(x, y), \]
where \( d \) is the distance from a point \((x, y)\) to the four neighboring points—north, south, east and west. Applying this rule to (11),
\[
\tilde{W}_n(x + d, y) + \tilde{W}_n(x - d, y) + \tilde{W}_n(x, y + d) + \tilde{W}_n(x, y - d)
- \left\{ \frac{4 + (2n-1)^2 \pi^2}{4} \left( \frac{d}{h} \right)^2 \cdot \frac{\mu}{A_1} + i \cdot \frac{2\omega \varrho d^2}{A_1} \cdot \sin \varphi \right\} \tilde{W}_n(x, y)
= \left( \frac{\partial D_n}{\partial x} + i \frac{\partial D_n}{\partial y} \right) \cdot \varrho \cdot \frac{d^2}{A_1} - \frac{2d^2}{h \cdot A_1} (\tau_x + i\tau_y).
\]
\[
(12)
\]
If the sum of the four values of $\overline{W}$ at four neighboring points is represented by $\Sigma \overline{W}$

$$\begin{align*}
-\left\{ 4 & + \frac{(2n-1)^2 \pi^2}{4} \cdot \frac{d^2}{h^2} \cdot \frac{\mu}{A_i} + i \frac{2 \omega \varrho d^2}{A_i} \cdot \sin \varphi \right\} \overline{W}_n \\
= & \left( \frac{\partial D_n}{\partial x} + i \frac{\partial D_n}{\partial y} \right) \cdot \frac{d^2}{A_i} - \frac{2 d^2}{h \cdot A_i} (\tau_x + i \tau_y) - \Sigma \overline{W}_n.
\end{align*}$$

(n = 1, 2, 3, \ldots)

(13)

Now it is easy to show that this equation can be solved by iteration if the boundary values are given. If $\overline{W}^{(m)}_n$ gives the $m$th approximation of $\overline{W}_n$ at $(x, y)$, then the $(m+1)$th approximation of $\overline{W}^{(m+1)}_n$ will be given by

$$\overline{W}^{(m+1)}_n = -\frac{\left( \frac{\partial D_n}{\partial x} + i \frac{\partial D_n}{\partial y} \right) x, y \cdot \frac{d^2}{A_i} - (\tau_x + i \tau_y) \cdot \frac{2 d^2}{h A_i} - \Sigma \overline{W}^{(m)}_n}{4 - \frac{(2n-1)^2 \pi^2}{4} \cdot \frac{d^2}{h^2} \cdot \frac{\mu}{A_i} + i \frac{2 \omega \varrho d^2}{A_i} \sin \varphi_{x, y}}.$$

(14)

The process must be started from points next to the boundaries where the values of $\overline{W}_n$ are given, so that the boundary values may be involved in $\Sigma \overline{W}_n$. The values which $\overline{W}_n$ should take at the boundaries, $G_n(0, y)$, $G_n(a, y)$, $G_n(x, 0)$ and $G_n(x, b)$, will be obtained by expanding the geostrophic currents at the vertical planes $x = 0, a$ and $y = 0, b$, also in similar series, or

$$\begin{align*}
G(0, y, z) & = \sum_n G_n(0, y) \cos \frac{(2n-1) \pi z}{2 h}, \\
G(a, y, z) & = \sum_n G_n(a, y) \cos \frac{(2n-1) \pi z}{2 h}, \\
G(x, 0, z) & = \sum_n G_n(x, 0) \cos \frac{(2n-1) \pi z}{2 h}, \\
G(x, b, z) & = \sum_n G_n(x, b) \cos \frac{(2n-1) \pi z}{2 h}.
\end{align*}$$

(15)

After a number of iteration processes, the values of $\overline{W}^{(m)}_n(x, y)$ will converge to a definite set of values, or
\[
\overline{W}_n(x, y) = \lim_{m \to \infty} \overline{W}^{(m)}_n(x, y).
\]

It is evident that the convergence of \( \overline{W}^{(m)}_n(x, y) \) will be obtained more quickly when the ratio

\[
\frac{\sum \overline{W}_n}{\left( \frac{\partial D_n}{\partial x} + i \frac{\partial D_n}{\partial y} \right)_{x, y} A_l} \cdot \frac{2 d^2}{h A_l} \cdot \sin \varphi_{x, y}
\]

is smaller, or if

\[
\left| \frac{(2n-1)^2 \pi^2}{4} \cdot \left( \frac{d^2}{h} \right) \cdot \frac{\mu}{A_l} + i \cdot \omega \cdot \frac{d^2}{A_l} \cdot \sin \varphi_{x, y} \right|
\]

is larger.

Thus the smaller the value of \( A_l \), the faster the convergence of the iteration. But we can show that successive approximations necessarily lead to a final convergence.

In this process, the boundary values will have less and less influence on \( \overline{W}_n \) as they recede from the boundaries. This permits the use of geostrophic current velocities for the boundary values, with only slight influence on the final results.

4. Zonal Flow: Mid-Pacific Equatorial Current System. Application of the present idea is best demonstrated in the case of zonal flow. In the equatorial region of the mid-Pacific, where the water moves mostly in an E-W direction, the dynamic equations will be reduced to two dimensions. Assume a zonal flow perpendicular to a meridional section extending from \( y = 0 \) to \( y = b \) and passing through the Equator (Fig. 1 B). Further, neglect the changes of \( u, v, \) and \( p \) in the \( x \)-direction. The dynamic equations (1), (2) or (7) then reduce to

\[
\frac{A_l}{\rho} \frac{\partial^2 \overline{W}}{\partial y^2} + \frac{\mu}{\rho} \frac{\partial^2 \overline{W}}{\partial z^2} - i \cdot 2 \omega \sin \varphi \overline{W} = i \frac{\partial D}{\partial y}.
\]

The boundary conditions to be satisfied at the surface and at the reference layer \( z = h \) remain the same, being given by

\[
z = 0: - \mu \frac{\partial \overline{W}}{\partial z} = \tau_x + i \tau_y
\]

and

\[
z = h: \overline{W} = 0.
\]
At both ends of this meridional section we have

\[ y = 0 : \quad \bar{W}(0, z) = G(0, z) = \sum_n G_n(0) \cos \left( \frac{2n-1}{2} \pi z \right), \]

and

\[ y = b : \quad \bar{W}(b, z) = G(b, z) = \sum_n G_n(b) \cos \left( \frac{2n-1}{2} \pi z \right), \]

where \( G(0, z) \) and \( G(b, z) \) are geostrophic currents at both ends of the profile, which can easily be computed from observed mass distribution.

Applying the same operation as that above, we have a similar expression to (14), or

\[
\bar{W}^{(m+1)}_n(y) = -\frac{i \left( \frac{\partial D_n}{\partial y} \right) \frac{\partial D^2}{A_l} \cdot (\tau_x + i \tau_y) - \frac{2 d^2}{h A_l} \left\{ \bar{W}^{(m)}_n(y + d) + \bar{W}^{(m)}_n(y - d) \right\}}{2 + \frac{2n-1}{4} \left( \frac{d}{h} \right)^2 \frac{\mu}{A_l} + i \frac{2 \omega \rho d^2}{A_l} \sin \varphi_y}. \tag{20}
\]
For starting values of 0th approximations, we employ

\[ W_n^{(0)}(y) = - \frac{i \left( \frac{\partial D_n}{\partial y} \right) \cdot \frac{d^2}{A_l} - (\tau_x + i \tau_y) \cdot \frac{2 d^2}{h A_l}}{2 + \frac{(2n - 1)^2 \pi^2}{4} \cdot \left( \frac{d}{h} \right)^2 \cdot \frac{\mu}{A_l} + i \frac{2 \omega_0 d^2}{A_l} \sin \varphi_y}, \]  

(21)

but it is possible to find better approximations by using the recurrence expression (20). In this case, values of \( W_n(y) \) at both ends of the profile are always given by

\[ W_n(0) = - \frac{i \left( \frac{\partial D_n}{\partial y} \right) \cdot \frac{d^2}{A_l} - (\tau_x + i \tau_y)_0 \cdot \frac{2 d^2}{h A_l}}{2 + \frac{(2n - 1)^2 \pi^2}{4} \cdot \left( \frac{d}{h} \right)^2 \cdot \frac{\mu}{A_l} + i \frac{2 \omega_0 d^2}{A_l} \sin \varphi_0}, \]  

and

\[ W_n(b) = - \frac{i \left( \frac{\partial D_n}{\partial y} \right)_b \cdot \frac{d^2}{A_l} - (\tau_x + i \tau_y)_b \cdot \frac{2 d^2}{h A_l}}{2 + \frac{(2n - 1)^2 \pi^2}{4} \cdot \left( \frac{d}{h} \right)^2 \cdot \frac{\mu}{A_l} + i \frac{2 \omega_0 d^2}{A_l} \sin \varphi_b}, \]  

(22)

5. Application to the Equatorial Undercurrent. Let us apply our theory to previous mid-Pacific observations (Cromwell, 1951); the profile extends over 20 stations approximately along 172° W from 5° S to 15° N (Sts. 26 to 6, M/V HUGH M. SMITH’s Cruise II, January–March 1950); the stations are spaced at intervals of 1° latitude, one of them always being located at the Equator. This gives \( d = 111.12 \) km.

The geopotential distances are given for 17 depths: 0, 10, 20, 30, 50, 75, 100, 150, 200, 250, 300, 400, 500, 600, 700, 800 and 1000 m, and the dynamic heights over the 1000 db-surface were calculated and analyzed according to the series:

\[ D(y, z) = \sum_n D_n(y) \cos \left( \frac{(2n - 1) \pi z}{2 h} \right). \]

It has been determined that \( D \) is well represented by this series if we use the 17 levels noted above. Then the meridional slopes can be calculated by taking successive differences of \( D \) and dividing them by \( d \).
As for the wind stress $\tau_x$ and $\tau_y$, these were calculated from Munk's (1947) formula based on wind observations taken on the same cruise. Strictly speaking, we probably should have used the synoptic distribution of wind stresses for the season in which the observations were taken, but it was more convenient to use the data taken on board simultaneously with the hydrographic casts.

It is most difficult to choose a reasonable set of values for $A_i$ and $\mu$. Previous authors found from different sources that $A_i$ varies from $10^7$ through $10^9$ and that $\mu/A_i$ is about $10^{-6}$. Thus it is sufficient to try the values $10^7$, $10^8$ and $10^9$ for $A_i$ in combination with the values of $\mu/A_i$ $10^{-7}$, $10^{-6}$, $10^{-5}$, etc.

6. Zonal Flow as Influenced by $A_i$ and $\mu$. Note that the calculated section of zonal flow depends largely on a combination of the values of $A_i$ and $\mu$, seven of which have been tried. Of these, four combinations $(A_i = 10^8$, $\mu = 10)$, $(A_i = 10^8$, $\mu = 10^3)$, $(A_i = 10^7$, $\mu = 1)$, and $(A_i = 10^7$, $\mu = 10^2)$ gave a calculated zonal velocity exceeding 300 cm/sec, which values are contrary to observation. Fig. 2, which shows the meridional variation of surface zonal velocities computed for several combinations of $A_i$ and $\mu$, demonstrates that the above-mentioned four pairs give quite unrealistic zonal velocities near the Equator. Therefore we shall discuss here the results of only three other combinations (1) $A_i = 10^9$ c.g.s., $\mu = 10^2$ c.g.s. (2) $A_i = 10^8$ c.g.s., $\mu = 10^2$ c.g.s. and (3) $A_i = 10^8$ c.g.s., $\mu = 10^3$ c.g.s. (Figs. 3, 4, and 5).

(1) Regarding the result for $A_i = 10^9$ c.g.s., $\mu = 10^2$ c.g.s., $\mu/A_i = 10^{-7}$, Figs. 3, 4 and 5 give the dynamic section of these isovels for the three combinations of $A_i$ and $\mu$ chosen above. $A_i = 10^9$ c.g.s., $\mu = 10^2$ c.g.s. gives a fairly reasonable result, with the distribution of isovels approximately similar to that observed by the Hawaiian oceanographers except that the absolute values of velocities seem a little lower and that the Equatorial Undercurrent is located somewhat more to the north than would be expected. The depth of the core of the Undercurrent also appears deeper than that observed, but this can be explained as a consequence of the assumption that $\mu$ is constant from surface to bottom. It appears that $A_i = 10^9$ c.g.s. is slightly large.

(2) The result for $A_i = 10^8$ c.g.s., $\mu = 10^2$ c.g.s., $\mu/A_i = 10^{-6}$ (Fig. 4) gives a strong current ($> 180$ cm/sec) at about $2^\circ$ S, a fact contrary to observation, and a faint eastward flow or a counter
Figure 2. Variation of surface zonal velocities across the meridional section of 172° W computed for several pairs of values of $A_I$ and $\mu$.

Figure 3. Isovels of zonal currents (cm/sec) across a dynamic section coincidental with the meridian of 172° W, computed from observations taken aboard the M/V Hugh M. Smith. $A_I = 10^9$ c.g.s. and $\mu = 10^3$ c.g.s. Shaded areas indicate eastward flow.
Figure 4. Isovels of zonal currents (cm/sec) across a dynamic section coincidental with the meridian of 172° W, computed from observations aboard the M/v HUGH M. SMITH, with $A_l = 10^4$ c.g.s. and $\mu = 10^3$ c.g.s. Shaded areas indicate eastward flow.

Figure 5. Isovels of zonal currents (cm/sec) across a dynamic section coincidental with the meridian of 172° W, computed from observations taken aboard the M/v HUGH M. SMITH, with $A_l = 10^3$ c.g.s. and $\mu = 10^3$ c.g.s. Shaded areas indicate eastward flow.
current is seen at Lat. 3° or 4° S. Here the Equatorial Undercurrent is shifted to the south, occupying just about the same position determined by previous observation.

(3) The result for $A_i = 10^8 \text{ c.g.s.}, \mu = 10^3 \text{ c.g.s.}, \mu/A_i = 10^{-5}$ (Fig. 5) gives an Equatorial Undercurrent that is too weak (about 15 cm/sec). Moreover, there are weak eastward flows in the surface layers south of the Equator. Thus $\mu = 10^3 \text{ c.g.s.}$ may be too large.

In the last two cases, the Equatorial Undercurrent rises up to the surface layers, but this of course is contrary to observation in this longitude.

These three sections, calculated for different combinations of $A_i$ and $\mu$, suggest that the most probable values of $A_i$ will be somewhere between $10^8$ and $10^9$ and that $\mu$ will be about $10^{-6}$. More reasonable results will be obtained by a more accurate choice of values for $A_i$ and $\mu$.

These calculations indicate that the Equatorial Undercurrent can be explained as a joint action of wind stresses on the surface, of planetary vorticity, and of a meridional pressure gradient.

CROMWELL, TOWNSEND

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