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EFFECT OF CORIOLIS FORCE ON EDGE WAVES
(I) INVESTIGATION OF THE NORMAL MODES

By

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ABSTRACT

Essentially two classes of free edge waves can exist on a sloping continental shelf in the presence of Coriolis force. For small longshore wave length, fundamental waves of the first class behave like Stokes edge waves. However, for great wave lengths (of several hundred kilometers or more) the characteristics of the first class are significantly altered. In the northern hemisphere the phase speed for waves moving to the right (facing shore from the sea) exceeds the speed for waves which move to the left. Also, the group velocity for a given edge wave mode has a finite upper limit. Waves of the second class are essentially quasigeostrophic boundary waves with very low frequency and, like Kelvin waves, move only to the left (again facing shore from the sea). Unlike Stokes edge waves, those of the quasigeostrophic class are associated with large vorticity. Examination of the formal solution for forced edge waves indicates that those of the second class may be excited significantly by a wind stress vortex. Also, in contrast to the conclusion of Greenspan (1956), it is proposed that a hurricane can effectively excite the higher order edge wave modes in addition to the fundamental if wind stress is considered.

INTRODUCTION

It was shown by Stokes (1846) that a sloping beach or continental shelf can act as a wave guide such as to allow for the possibility of surface gravity waves which travel along shore with an amplitude which decays exponentially with distance from shore (Fig. 1). Ursell (1952) has shown that the Stokes wave is the fundamental mode of a discrete spectrum of edge wave modes which are possible. The higher order modes are characterized by an integer number of nodal lines parallel to shore.

In the Stokes-Ursell theory, no restrictions are imposed upon the vertical pressure gradient, and results are therefore applicable for a beach of any uniform slope adjacent to a straight coastline. For suitably small bottom slope, Eckart (1951) has shown that it is possible to introduce the hydrostatic pressure approximation (as is usually

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2 Following common usage, “higher order” here implies higher than the fundamental or first order.
done in the theory of long waves). This approximation considerably simplifies the mathematical treatment, and Greenspan (1956) has applied this long wave theory in a detailed investigation of pressure-induced edge waves on the continental shelf. In the development undertaken by Greenspan, the influence of the earth's rotation is ignored as an understandable first approximation. It is probable, however, that resurgences investigated by Greenspan as well as by Munk, *et al.* (1956) represent waves of such lengths that Coriolis force should play a significant role.

The primary object of the present paper is to investigate the role of Coriolis force in respect to the characteristics of free edge waves. However, some consideration is given to the problem of *generation* of edge waves. The theory is confined to the case of small slope for which the hydrostatic pressure approximation is applicable. Furthermore, as in previous papers on this subject, the influence of bottom friction is ignored.

It is found that characteristics of the conventional edge waves, in the presence of the earth's rotation, are significantly modified for very long waves, particularly for the fundamental mode. In addition, a quasigeostrophic class of edge waves with relatively large vorticity is possible and bears some resemblance to the Kelvin boundary wave. The quasigeostrophic edge waves are evidently the counterpart of

![Figure 1. Schematic of the fundamental edge wave mode showing contours of water level (full lines for positive \( \eta \), dashed lines for negative \( \eta \)).](image_url)
the steady vortex flow which could exist in the absence of the earth’s rotation.

The question of excitation of the different modes is discussed from a general point of view in an attempt to arrive at some useful criteria. In another paper (Kajiura, 1958), the particular case of excitation of the fundamental edge wave mode on a rotating earth is discussed in detail, and numerical examples are given to illustrate the influence of Coriolis force.

LIST OF SYMBOLS

The symbols which are listed below occur frequently in the discussion; other symbols are introduced in the text as required:

- $A$: amplitude at $x = 0$
- $A_{jn}(k, t)$: complex amplitude spectrum, eq. (68) or (101); dimension $[L^2]$
- $C_{jn}(k)$: longshore phase velocity $(-\omega_{jn}/k)$
- $D_{jn}(x, y; k)$: horizontal divergence of the normal mode currents
- $\bar{D}_{jn}(k)$: amplitude of $D_{jn}$ at $x = 0$
- $e$: 2.71828 . . .
- $F_x, F_y$: $x$ and $y$ components of the vector-forcing function $F(x, y, t)$.
- $f$: Coriolis parameter $2\Omega \sin \phi$, where $\Omega$ is the angular speed of the earth, $\phi$ the latitude.
- $G_{jn}(k)$: longshore group velocity $(-d\omega_{jn}/dk)$
- $g$: acceleration due to gravity
- $H_{jn}(x, y; k)$: normal mode elevation function for $A = 1$
- $h$: still-water depth $(sx)$
- $i$: $\sqrt{-1}$
- $j$: denotes mode: 1, 2, 3
- $k$: longshore wave number $(2\pi/\lambda)$
- $L_n(z)$: Laguerre polynomial, eq. (26)
- $L'_n(z)$: $dL_n(z)/dz$
- $M_{jn}(k, t)$: amplitude excitation, $dA_{jn}(k, t)/dt$
- $N_{jn}(k)$: norm of the eigenfunctions, eq. (64); dimension $[L]$
- $n$: order designation: 0, 1, 2, 3 . . . (number of nodal lines parallel to shore)
- $P(x, y, t)$: atmospheric pressure at sea level
- $P'(x, y, t)$: departure of $P$ from normal
- $p_{jn}(k)$: weight function, $[2 |k| N_{jn}(k)]^{-1}$
- $Q_x(x, y, t)$: $x$ component of volume transport per unit width
- $Q_y(x, y, t)$: $y$ component of volume transport per unit width
- $q_n(z)$: Laguerre orthonormal function, eq. (25)
The superscript * is employed to denote the complex conjugate of the function indicated. The formal designation of functional dependence is omitted in those cases where clarity is not imperiled. In the first part of the development the subscripts \( j \), \( n \) do not appear in respect to the normal mode functions.

**FORMULATION OF THE PROBLEM**

1. **The Basic Equations.** We will confine our attention to undamped waves of small amplitude and of sufficient length such that linearized equations of motion with the hydrostatic pressure approximation are applicable. It will be assumed also that the density of the water is uniform. The vertically integrated, linearized equations of motion for a forced, barotropic inertiogravitational disturbance are then

\[
\frac{\partial Q_x}{\partial t} - f Q_y + g h \frac{\partial \eta}{\partial x} = F_z, \tag{1}
\]
Reid: Effect of Coriolis Force

\[ fQ_x + \frac{\partial Q_v}{\partial t} + gh \frac{\partial \eta}{\partial y} = F_v, \]  
(2)

and the equation of continuity is

\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_v}{\partial y} + \frac{\partial \eta}{\partial t} = 0, \]  
(3)

neglecting precipitation or evaporation at the sea surface. The Coriolis parameter \( f \) (positive in the northern hemisphere) will be considered as a constant in the present analysis, hence planetary type waves are excluded from the analysis. The forcing functions \( F_x \) and \( F_v \) are presumed to be specified in terms of space and time variables. They are physically determined by the horizontal gradient of the atmospheric pressure at sea level, the surface wind stress, and the depth of water [see relation (79)]. It is recognized that omission of bottom stress may be a rather severe restriction in view of (4) and may deserve further attention.

The \( x, y \)-axes constitute a right-hand coordinate system, with the \( x \)-axis directed seaward and the \( y \)-axis directed along shore and coincident with the undisturbed position of the water’s edge (presumed straight). It is assumed that the increase in depth is directly proportional to the distance from shore and that the sea is considered semi-infinite; thus

\[ h = sx, \]  
(4)

the bottom slope \( s \) being considered very small \((s \ll 1)\). This simplification places definite restrictions on the applicability of the theory to waves whose effective length normal to shore is very large (greater than the actual width of the continental shelf).

To complete the formulation of the problem we suppose that \( \eta, Q_x \) and \( Q_v \) are specified as functions of \( x \) and \( y \) at \( t = 0 \). Finally as boundary conditions, we take \( \eta \) as finite at shore and

\[ Q_x = 0 \quad \text{at} \quad x = 0 \quad \text{for all} \quad y \quad \text{and} \quad t; \]  
(5)

and

\[ Q_x, Q_v, \eta \to 0 \quad \text{as} \quad x, |y| \to \infty \quad \text{for all} \quad \text{finite} \quad t. \]  
(6)

Condition (6) is possible only if the vector-forcing function \( \mathbf{F} \) is localized in the vicinity of the coastline, such that \( \mathbf{F} \to 0 \) as \( x, |y| \to \infty \), and if no initial disturbance exists at infinity. The periodic tides and the steady oceanic-scale circulation are therefore excluded at the outset. For a transient atmospheric disturbance in which \( \mathbf{F} \to 0 \) as \( t \to \infty \), presumably a finite amount of energy is added to the water for \( t > 0 \),
and this energy is ultimately transferred to infinite $y$ by radiation as $t \to \infty$. However, for finite $t$ we will consider that (6) holds.

The condition of finite $\eta$ at shore automatically excludes a certain class of solutions which have logarithmic singularities at $x = 0$ (Stoker, 1957: 71–72). This is certainly reasonable in view of the linearized nature of the problem. However, despite the lack of physical significance of such solutions, one is tempted to regard these as the linear approximation of unstable disturbances at the water's edge. Stoker points out that exclusion of such solutions precludes the possibility of an energy sink at shore; consequently it excludes the possibility of wave energy progressing towards shore at $x = \infty$. From a practical standpoint this implies that ordinary sea and swell are neglected in the present problem, the attention being directed entirely to the question of long edge waves.

2. Associated Normal Mode Problem. The expressions for $Q_z, Q_y$ and $\eta$ which satisfy the above relations with $F_z$ and $F_y$ set equal to zero will be designated as $U, V$ and $H$ respectively. These free wave functions therefore must satisfy the differential equations

$$\frac{\partial U}{\partial t} - fV + gsx \frac{\partial H}{\partial x} = 0, \quad (7)$$

$$fU + \frac{\partial V}{\partial t} + gsx \frac{\partial H}{\partial y} = 0, \quad (8)$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial H}{\partial t} = 0, \quad (9)$$

subject to the conditions that $H$ is finite at shore and that

$$U = 0 \quad \text{at } x = 0, \quad (10)$$

$$U, V, H \to 0 \quad \text{as } x \to \infty. \quad (11)$$

However, we make no stipulation in regard to initial conditions or to the condition for $|y| \to \infty$.

In order to establish the normal mode functions, we first introduce the transformations

$$U = X(x; k, \omega) \exp [i(ky + \omega t)], \quad (12)$$

$$V = Y(x; k, \omega) \exp [i(ky + \omega t)], \quad (13)$$

$$H = Z(x; k, \omega) \exp [i(ky + \omega t)]. \quad (14)$$
These expressions will satisfy (7) to (11) provided that the functions $X$, $Y$ and $Z$ satisfy the relations

$$i\omega X - fY + gsx \frac{dZ}{dx} = 0, \quad (15)$$

$$fX + i\omega Y + igsxkZ = 0, \quad (16)$$

$$\frac{dX}{dx} + ikY + i\omega Z = 0, \quad (17)$$

and

$$X = 0 \quad \text{at } x = 0, \quad (18)$$

$$X, Y, Z \to 0 \quad \text{as } x \to \infty, \quad (19)$$

where $Z$ is finite at shore.

The elementary free waves defined by (12) to (19) are simple harmonic progressive waves of longshore length $2\pi/k$, period $2\pi/\omega$, and longshore phase speed $-\omega/k$. The wave number $k$ is to be regarded as real but is otherwise arbitrary; consequently there exists a continuous spectrum of normal mode solutions in respect to $k$. However, for a given $k$, there are only certain admissible functions $X, Y, Z$ which in turn are associated with particular values of $\omega$. These represent respectively the eigenfunctions and characteristic frequencies of the system (15) to (19) for a given $k$.

From (15) and (16) we find that

$$X = \frac{igsx}{(\omega^2 - f^2)} \left[ \omega \frac{dZ}{dx} + fkZ \right], \quad (20)$$

$$Y = \frac{-gsx}{(\omega^2 - f^2)} \left[ f \frac{dZ}{dx} + \omega kZ \right]. \quad (21)$$

Inserting these relations in (17) yields

$$\omega s \left[ \frac{d}{dx} \left( x \frac{dZ}{dx} \right) + (\kappa - k^2 x)Z \right] = 0, \quad (22)$$

where

$$\kappa = \frac{\omega^2 - f^2}{gs} + \frac{fk}{\omega}. \quad (23)$$

The set of normal mode solutions, containing the eigenfunctions and characteristic frequencies, forms an essential part of the general solution of the forced and/or initial value problem specified by (1) to
(6). In fact this constitutes the primary justification for the investigation of the normal modes. It may be noted immediately from (20) that condition (18) and hence (10) are automatically satisfied if \( Z \) and its derivative are finite at shore. Consequently the addition of Coriolis force to the edge wave problem does not lead to the complication of a mixed boundary condition which is encountered in problems of very long surges incident upon a vertical barrier (Crease, 1956).

3. Standing “Shelf” Modes. Our main interest centers on the edge wave modes for which \( \omega \) and \( k \) differ from zero, since it is only this type which can be effectively excited by a localized atmospheric disturbance. However, there are several nontrivial solutions of (15) to (17) for \( k = 0 \), with finite \( Z \) at shore, which might be mentioned in passing. These modes represent either standing or stationary wave patterns on the sloping shelf with uniform flow and water level along shore.

The case of \( k = \omega = 0 \) represents a steady geostrophic mode with \( H \) contours and with flow parallel to shore. Any transverse distribution \( Z(x) \) is admissible subject to condition (19). For \( k = 0 \) and \( \omega > f \), (22) admits a solution with finite \( Z(0) \) of the form

\[
Z(x) = AJ_0(2\sqrt{kx}),
\]

where \( J_0 \) is the Bessel function of the first kind, order zero, and where \( A \) is an arbitrary amplitude. The disturbance represents a standing shelf oscillation which is independent of \( y \) but which has an oscillatory component of flow parallel to shore as well as transverse to shore. The period of these waves is less than 12 pendulum hours \((2\pi/f)\). It is conceivable that this mode is of importance in the study of co-oscillating tides on the continental shelf.

For \( k = 0 \) and \( \omega = f \), (15) to (17) admit a peculiar type of inertial motion with periodic but spatially uniform surface displacement. This is a degenerate case of (24), and as we shall see it is also a limiting case of higher order edge wave modes. Relations (20) and (21) lead to indeterminate forms in this case, but it is readily established from (16) and (17) that both \( X \) and \( Y \) have the magnitude \( Af \), hence the horizontal trajectories of the particles are circular as in simple inertial motion. However, in view of its spatially uniform properties, it is evident that this mode does not satisfy (19) unless \( A = 0 \), hence it cannot be excited by a localized atmospheric disturbance.
**EDGE WAVE MODES**

1. **Edge Wave Eigenfunctions.** For nonvanishing $\omega$ and $k$ it can be shown (Eckart, 1951: 48–50) that the only solutions of (22) which lead to finite $Z$ for all positive $x$ are the functions $A q_n (2 |k| x)$, where $A$ is an arbitrary amplitude and $q_n(z)$ is the Laguerre function of order $n$:

$$q_n(z) = \frac{e^{-z^2/2}}{n!} L_n(z), \quad (25)$$

where $n = 0, 1, 2, 3 \ldots$ and

$$L_n(z) = n! \left[ 1 - nz + \frac{n(n - 1)}{(2!)^2} z^2 - \frac{n(n - 1)(n - 2)}{(3!)^2} z^3 + \ldots \frac{(-1)^r n(n - 1) \ldots (n - r + 1)}{(r!)^2} z^r + \ldots \right], \quad (26)$$

Note that for $z = 0$, $L_n = n!$; consequently $q_n(0) = 1$ for all $n$.

The Laguerre polynomials, $L_n(z)$, satisfy the differential equation

$$z \frac{d^2 L_n}{dz^2} + (1 - z) \frac{dL_n}{dz} + nL_n = 0. \quad (27)$$

If the function $q_n (2 |k| x)$ is substituted for $Z$ in (22), the latter reduces to (27) provided that

$$\kappa = (2n + 1) |k|. \quad (28)$$

This is the characteristic equation to be satisfied by normal modes and it yields the allowable values of $\kappa$ and hence $\omega$ for a given longshore wave number $k$. Note that there is no effect of Coriolis force on the surface form of the edge wave modes. However, the flow associated with these waves is definitely influenced by Coriolis force, as is evident from (20) and (21). The fundamental waves are characterized by an exponentially decaying amplitude of the surface with increasing $x$ (Fig. 1), while the higher order modes are characterized by $n$ discrete nodal lines offshore; beyond the $n$th nodal line the amplitude decays monotonically as $(2 |k| x)^n e^{-|k| x}$. Tables of the Laguerre functions through order 7 are given by Eckart (1934).

The Laguerre functions form an orthonormal set in the sense that

$$\int_0^\infty q_n(z) q_m(z) \, dz = \begin{cases} 0 & \text{for } n \neq m, \\ 1 & \text{for } n = m, \end{cases} \quad (29)$$

where $m$ is a positive integer or zero (Courant and Hilbert, 1931: 79–82). Later on we shall note a more general orthogonality property.
of normal mode functions which is useful in establishing the formal solution of the forced wave problem.

2. Edge Wave Frequencies. By employing (23) we may write the characteristic relation (28) in the form

\[ \omega^3 - [f^2 + (2n + 1)gs|k|]\omega + gskf = 0. \]  

(30)

For any given order, \( n \), and wave number, \( k \), there are three allowable values of \( \omega \). The admissible frequency functions will be denoted by \( \omega_{jn}(k) \), where \( j = 1, 2, 3 \) denotes the three different modes corresponding to each order \( n \). An examination of the cubic equation (30) reveals that the roots \( \omega_{jn}(k) \) are real and have the following general properties:

\[ \omega_{jn}(-k) = -\omega_{jn}(k), \]  

(31)

\[ \omega_{1,n} + \omega_{2,n} + \omega_{3,n} = 0, \]  

(32)

and

\[ \omega_{1,n}\omega_{2,n}\omega_{3,n} = -gskf. \]  

(33)

It is evident from (32) and (33) that two of the roots are positive and one is negative for each mode, if \( kf \) is positive. Furthermore, the \( \omega_{jn}(k) \) functions are antisymmetric with respect to \( k \), which indicates that the phase velocities \( -\omega_{nj}/k \) are independent of the sign of \( k \). Accordingly we can confine our attention to the case of positive \( k \) alone. Finally we will adopt \( \omega_{1,n} \) as the negative root, corresponding to waves which move in the positive \( y \) direction.

For the fundamental mode \( (n = 0) \), relation (30) can be factored such that for positive \( k \):

\[ (\omega - f)[\omega(\omega + f) - gsk] = 0. \]  

(34)

Consequently

\[ \omega_{1,0} = -\frac{1}{2}f - \sqrt{gsk + \frac{1}{4}f^2}, \]  

(35)

\[ \omega_{2,0} = -\frac{1}{2}f + \sqrt{gsk + \frac{1}{4}f^2}, \]  

(36)

\[ \omega_{3,0} = f. \]  

(37)

Relations (20) and (21) are indeterminate for the case of \( \omega = f \), and an analysis of (15) to (17) for this case indicates that the only meaningful solutions satisfying conditions (18) and (19) occur for \( k = 0 \) and \( k = 2f/gs \). The first of these cases, the standing inertial mode discussed earlier, is a limiting case of the higher order waves of mode \( j = 2 \); the other is a special case of the fundamental waves of mode \( j = 2 \). Thus
for the fundamental edge waves, only the modes $j = 1, 2$ need be considered.

Note that, for no rotation of the earth, $\omega = \pm \sqrt{gsk}$ for the fundamental mode. This corresponds to the frequency of Stokes edge waves for sufficiently small bottom slope. However, in the presence of rotation, the frequencies $\omega_{1,0}$ and $\omega_{2,0}$ differ in magnitude by $f$; hence the phase speeds for waves moving to right or left along shore differ by $f/k$ for the fundamental mode.

It is usually convenient to express the frequency in terms of wave period $T_{jn} = 2\pi/|\omega_{jn}|$. The periods corresponding to $\omega_{1,0}$ and $\omega_{2,0}$, as evaluated from (35) and (36) for several different values of longshore wave length ($\lambda$), are given in Table I based upon a latitude of $45^\circ$ and $s = 10^{-3}$ (about 1 fathom per nautical mile). The period as computed for the case of no rotation, $T_0 = (2\pi/\sqrt{gsk})$, is tabulated for comparison. The last column gives the percent difference between periods of waves moving to right and left. As we should expect, the greatest difference occurs for the longest waves, those for which $T_0$ approaches or exceeds the pendulum period $(2\pi/f)$; the influence of the earth's rotation is essentially negligible for periods less than one hour.

As applied to the east coast of the United States, taking $s = 10^{-3}$, the fundamental edge waves of 1000 km wave length would have a period of 5.70 hours if traveling north, or a period of 8.63 hours if traveling south. Munk, et. al. (1956) have suggested that edge waves of about 5 to 6 hour period can be generated by hurricanes moving northward along the east coast. The present analysis indicates that the Coriolis force can have a significant influence on the characteristics of fundamental edge waves of this period.

| TABLE 1.—PERIODS OF THE FUNDAMENTAL MODE ($n = 0$) FOR $45^\circ$ LATITUDE |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $\lambda$ (km)              | $T_{1,0}$ (hrs)             | $T_{2,0}$ (hrs)             | $T_0$ (hrs)                 | $T_{2,0} - T_{1,0}$ (%)    |
| $s = 10^{-3}$                | with Rotation              | No Rotation                |                             |                              |
| 10                          | 0.690                      | 0.718                      | 0.703                       | 4                            |
| 100                         | 2.08                       | 2.37                       | 2.22                        | 13                           |
| 1000                        | 5.70                       | 8.63                       | 7.03                        | 42                           |
| 10,000                      | 12.0                       | 41.2                       | 22.2                        | 132                          |

The general roots of (30) for the higher order edge waves ($n \geq 1$) are given by

$$\omega_{jn} = -\frac{2}{\sqrt{3}}f \sqrt{1 + (2n + 1)gsk/f^2} \cos [\beta + (j - 1)2\pi/3],$$ (38)
where
\[ \cos 3\beta = \frac{3}{2} \sqrt{3} \frac{gsk}{f^2} [1 + (2n + 1)gsk/f^2]^{-3/2}. \] (39)

The asymptotic behavior of the functions \( \omega_{j,n}(k) \) for very small \( k \) and very large \( k \) is indicated in Table II for all \( n \). The frequency functions are also illustrated in Figs. 2 and 3. In the logarithmic plot (Fig. 3), only modes \( j = 2, 3 \) are shown; \( \omega_{1,n} \) can be computed readily by using (32).

The function \[ |\omega_0/f| = \sqrt{gsk/f^2} \] representing the relative frequency of Stokes edge wave is shown by the dashed curve in both figures, for comparison. It is of interest to note that the fundamental mode for

![Figure 2. Relative values of the characteristic frequency functions, \( \omega_{j,n}(k)/f \), versus relative wave number, \( gsk/f^2 \) (dimensionless); the different orders \( n \) are indicated by the numbers in circles; the mode \( j = 1 \) represents waves progressing in the positive \( y \) direction (e.g., northward along the east coast of a continent in the northern hemisphere). Auxiliary scales for period and wave length corresponding to \( 45^\circ \) latitude and \( s = 10^{-4} \) are shown for convenience.](image-url)
Figure 3. Relative frequencies $\omega_j(k)/f$ versus $gsk/f^2$ for modes 2 and 3 on a log-log plot; this figure emphasizes the peculiar behavior of the fundamental wave ($j = 2$) relative to the higher order waves of modes 2 and 3. Note that modes 2 and 3 correspond to waves progressing in the negative $y$ direction.

$j = 2$ is really in a class apart from the other waves; for large $k$ it behaves like Stokes wave but for very small $k$ it behaves like the higher order quasigeostrophic modes (see Fig. 3 or Table II). On the other hand, the fundamental mode for $j = 1$ behaves like an inertial oscillation with frequency $f$ for very small $k$.

For all $n$, the modes $j = 1, 2$ behave like purely gravitational edge waves if $k$ is sufficiently large. Ursell (1952) obtained the result

$$\omega = \pm \sqrt{gk \sin (2n + 1)\alpha},$$  \hspace{1cm} (40)

where $\alpha$ is the angular inclination of the sea bed in radians. If $(2n + 1)\alpha << 1$, then (40) reduces to the asymptotic expressions given in Table II for $j = 1, 2$ and large $k$, since in this case $\alpha$ and $s$ are essentially the same. An interesting feature of Ursell’s analysis is that it predicts an upper limit on $n$, corresponding to $(2n + 1)\alpha \simeq \pi/2$. However, even for $s = 10^{-2}$ the maximum $n$ is well over 70.

The mode $j = 3$, which exists for the higher order waves ($n \geq 1$), represents an entirely different class of waves. The frequency of these waves is very small ($< f$) for all values of $k$ and, as will be shown presently, they are characterized by quasigeostrophic flow and rela-
tively large vorticity. The counterpart of this class of disturbance in the case of $f = 0$ is a steady vortex mode with no surface distortion. The quasigeostrophic waves ($j = 3$) propagate only in the negative $y$ direction in the northern hemisphere (for example, to the south along the east coast of the United States). In this respect they are analogous to Kelvin boundary waves (Proudman, 1953: 253–255).

<table>
<thead>
<tr>
<th>Condition</th>
<th>Order</th>
<th>Mode</th>
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<tbody>
<tr>
<td>$k \ll \frac{f^2}{(2n + 1)gs}$</td>
<td>$n = 0$</td>
<td>$j = 1$</td>
</tr>
<tr>
<td>$k \gg \frac{f^2}{(2n + 1)gs}$</td>
<td>$n &gt; 1$</td>
<td>$j = 2$</td>
</tr>
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3. Group Velocity and Dispersive Properties. In a study of non-periodic free waves emerging from some initially localized disturbance (e.g., the Cauchy-Poisson problem; Lamb, 1932: 384–391), the group velocity, $-d\omega / dk$, is of importance since it is essentially a measure of the rate of propagation of a given segment of the energy spectrum. The dependence of group velocity on $k$ consequently governs the dispersion or resolution of the resulting wave train. Using (30), the following formula is obtained for the group velocity for $k > 0$ and $n \geq 1$:

$$G_{jn}(k) = -\frac{d\omega_{jn}}{dk} = \frac{-gs[2n + 1]\omega_{jn} - f}{[3\omega_{jn}^2 - f^2 - (2n + 1)gs]}$$

(41)

For the special case $n = 0$, (35) and (36) yield directly

$$G_{j,0}(k) = \pm \frac{gs}{\sqrt{4gs^2 + f^2}}$$

(42)

for $j = 1, 2$ respectively. Thus the group velocity of the two fundamental modes are equal in magnitude even though the phase speeds differ.

It follows from (42) that the group velocity of the fundamental edge waves cannot exceed $gs/f$. For the higher order modes, the maximum group velocity is $(n + 1)gs/f$, corresponding to the $j = 1$ modes for $k \rightarrow 0$. For $s = 10^{-3}$, the maximum group velocity of the fundamental waves is about 100 m/sec at midlatitudes and corresponds to about half the speed of a tsunami in the open ocean.
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<thead>
<tr>
<th>Condition</th>
<th>Condition</th>
<th>Order</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k \ll \frac{f^2}{(2n + 1)g_s} )</td>
<td>( n = 0 )</td>
<td>( \frac{g_s}{f} \left(1 - 2 \frac{gsk}{f^2}\right) )</td>
<td>( j = 1 )</td>
</tr>
<tr>
<td>( n &gt; 1 )</td>
<td>( (n + 1) \frac{g_s}{f} \left[1 - (n + 2) \frac{gsk}{f^2}\right] )</td>
<td>( -\frac{g_s}{f} \left(1 - 2 \frac{gsk}{f^2}\right) )</td>
<td>( j = 2 )</td>
</tr>
<tr>
<td>( k \gg \frac{f^2}{(2n + 1)g_s} )</td>
<td>( n = 0 )</td>
<td>( \frac{1}{2} \sqrt{\frac{g_s}{k}} )</td>
<td>( (0) )</td>
</tr>
<tr>
<td>( n &gt; 1 )</td>
<td>( \frac{1}{2} \sqrt{\frac{(2n + 1)g_s}{k}} )</td>
<td>( -\frac{1}{2} \sqrt{\frac{g_s}{k}} )</td>
<td>( (0) )</td>
</tr>
</tbody>
</table>

\[ g_{3n}(k) \]
The asymptotic expressions for group velocity are listed in Table III, and a plot of the group velocity as a function of $k$ for $n = 0, 1, 2$ and $j = 2, 3$ is shown in Fig. 4. It is evident that the group velocity of the quasigeostrophic mode is practically zero for $k > 2f^2/gs$. This implies that the propagation of wave energy away from an initially localized disturbance is principally due to the ordinary edge wave modes $j = 1, 2$, unless the wave spectrum contains a preponderance of energy in the range $0 < k \ll 2f^2/gs$.

![Figure 4. Relative group velocity $|G_{jn}/gs|$ versus $gsk/f^2$ for modes 2 and 3 and orders 0, 1, 2. Note that $G_{1,n}$ is readily obtained by the expression $|G_{1,n} + \Sigma G_{1,n}|$.](image)

**KINEMATIC BEHAVIOUR OF THE EDGE WAVE MODES**

1. **Character of the Fluid Flow.** Hereafter we will employ the following notation for the normal mode flow and elevation functions:

$$U_{jn}(x, y, t; k) = X_{jn}(kx) \exp [iky + i\omega_{jn}(k)t], \quad (43)$$

$$V_{jn}(x, y, t; k) = Y_{jn}(kx) \exp [iky + i\omega_{jn}(k)t], \quad (44)$$

$$H_{jn}(x, y, t; k) = Z_{n}(kx) \exp [iky + i\omega_{jn}(k)t]. \quad (45)$$

For each of the three frequency modes ($j = 1, 2, 3$) there is a corresponding set of eigenfunctions; however, all three modes have $Z_n(kx)$ in common.

For convenience we will take the amplitude of the surface disturbance at $x = 0$ as unity, so that $Z_n(kx) = q_n(2|k|x)$. By employ-
ing the appropriate recursion formulas for the Laguerre polynomials (Courant and Hilbert, 1931: 79–80), we then find from (20) and (21) for \( k \geq 0, n \geq 1 \):

\[
X_{jn}(kx) = -ig s \left\{ \frac{kx Z_n(kx)}{(\omega_{jn} + f)} - \frac{n\omega_{jn}}{\left( \omega_{jn}^2 - f^2 \right)} [Z_n(kx) - Z_{n-1}(kx)] \right\},
\]

\[
Y_{jn}(kx) = -gs \left\{ \frac{kx Z_n(kx)}{(\omega_{jn} + f)} + \frac{n f}{\left( \omega_{jn}^2 - f^2 \right)} [Z_n(kx) - Z_{n-1}(kx)] \right\},
\]

\[
Z_n(kx) = \frac{e^{-kz}}{n!} L_n(2kx).
\]

In the case of the fundamental mode for \( k \geq 0 \) and \( j = 1, 2 \) the following eigenfunctions apply:

\[
X_{j,0}(kx) = -\frac{igskx}{\left( \omega_{j,0} + f \right)} e^{-kx},
\]

\[
Y_{j,0}(kx) = -\frac{gskx}{\left( \omega_{j,0} + f \right)} e^{-kx},
\]

and

\[
Z_0(kx) = e^{-kz}.
\]

It can be shown, from a separate analysis for negative \( k \), that \( X_{jn} \) is an odd function with respect to \( k \) and that \( Y_{jn} \) and \( Z_n \) are even.

In general the component of flow \( U_{jn} \) is \( \pi/2 \) radians out of phase with the \( V_{jn} \) component and the latter is in phase with the water level. Furthermore, the amplitudes of \( U_{jn} \) and \( V_{jn} \) differ in magnitude for \( n \geq 1 \); thus, in the Lagrangian system, the paths of the water particles describe ellipses in a nearly horizontal plane. However, in the case of the fundamental mode, we note from (49) and (50) that the \( x \) and \( y \) components of flow are of equal amplitude, and hence the horizontal projections of the particle orbits are circular. The orbit radius, \( r \), at a given mean position offshore, can be evaluated by integrating \( U_{j,0}/h \) or \( V_{j,0}/h \) with respect to \( t \) and by using (34); the result is

\[
r = \frac{1}{s} e^{-|k|} x
\]

for unit amplitude at shore.

Both components of volume transport vanish at the water's edge. However, the components of fluid velocity \( U_{jn}/h \) and \( V_{jn}/h \) do not vanish at this position. It is simple to evaluate these components for the fundamental waves, but in the case of the higher order waves we
must consider the limit of \((Z_n - Z_{n-1})/kx\) as \(kx \to 0\). Making use of (30), the result is for all \(j, n, k\):

\[
\left[ \frac{U_{jn}}{h} \right]_{x=0} = -i \frac{\omega_{jn}}{s} e^{i(ky + \omega_{jn}t)},
\]

(53)

\[
\left[ \frac{V_{jn}}{h} \right]_{x=0} = \frac{\omega_{jn}}{s} \left[ \frac{(2n + 1)f}{(2n + 1)\omega_{jn} - f} \right] e^{i(ky + \omega_{jn}t)}.
\]

(54)

As a check, we note that the amplitude of horizontal excursion of the water's edge, as obtained from the integral of (53) with respect to \(t\), is simply \(1/s\). This must be the case in the linear approximation with unit amplitude of \(Z_n\) at shore. The longshore excursion of the fluid particles at the water's edge has an amplitude \(\leq 1/s\) for modes \(j = 1, 2\), but for \(j = 3\) the amplitude of longshore excursion is much greater than \(1/s\).

An interesting feature worth noting is that, although the amplitudes of flow are different for the \(j = 1\) and \(j = 2\) modes of the fundamental edge wave on a rotating earth, the orbit radii are the same at a given position offshore. From (49) and (50), using (35) and (36), it follows that the fundamental waves which move in the positive \(y\) direction \((j = 1)\) are associated with a greater magnitude of flow than the fundamental waves which move in the opposite sense, assuming the same wave length and water level amplitude at shore for both.

In the case of the \(j = 3\) modes, the frequency is much less than \(f\) for all \(k\), and since \(|dZ_n/dx|\) is of the order \(|(2n + 1)kZ_n|\), it follows from (21) that

\[
Y_{3,n}(kx) \sim \frac{g^3}{f^2} \frac{dZ_n}{dx}.
\]

(55)

Thus the longshore flow is in quasigeostrophic balance with the offshore slope of the sea surface for any \(k\) associated with this mode. The fundamental waves for mode \(j = 2\) also have this property when \(k\) is sufficiently small \((\ll f^2/gs)\). It is of interest that the quasigeostrophic character applies only to waves propagating in the negative \(y\) sense.

2. Vorticity and Divergence. We can gain a greater appreciation of the distinction in kinematic behavior of various edge wave modes by investigating the vorticity and divergence of these waves. It may be remarked that one of the outstanding features of Stokes edge waves, which sets them apart from all other surface gravity waves, is that they are horizontally divergenceless. This unusual property is possible only in the presence of the sloping sea bed; the variations in the surface elevation can be regarded, in the Lagrangian sense, as being associated
with the onshore or offshore displacements of a column of sea water (of fixed height) along the sloping bed. Thus the rate of rise or fall of sea level is directly proportional to the onshore or offshore component of horizontal velocity of the water for a given bottom slope.

The vertical component of vorticity in Stokes edge waves is also zero. The irrotational and horizontally divergenceless property of the fundamental mode is not modified by the presence of the earth's rotation (the vorticity relative to the earth being zero). However, the higher order waves of both classes do possess vorticity and horizontal divergence. As we shall see presently, this property is important in determining the excitation of higher order edge waves by wind stress.

The vertical component of relative vorticity, \( \xi \), and the horizontal divergence, \( D \), are defined respectively by

\[
\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\]

and

\[
D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},
\]

where \( u = U/h \) and \( v = V/h \), representing the vertically averaged components of fluid velocity. Using (43), (44) and (46) to (48), it can be shown that for any normal mode

\[
\xi_{jn}(x, y, t; k) = \frac{4gjk^2}{(\alpha^2_{jn} - f^2)} \frac{e^{-|k|z}}{n!} \left[ \frac{nL_n(2|k|x) + L'_n(2|k|x)}{(2|k|x)} \right] \times \exp [iky + i\omega_{jn}(k)t]
\]

and

\[
D_{jn}(x, y, t; k) = -i \frac{\omega_{jn}}{f} \xi_{jn}(x, y, t; k),
\]

where

\[
L'_n(z) = dL_n(z)/dz.
\]

Note first of all that, for the fundamental mode, \( L_0(z) \) is unity for all \( z \), and \( L'_0(z) \) vanishes identically; consequently

\[
\xi_{j,0}(x, y, t; k) = D_{j,0}(x, y, t; k) = 0,
\]

as in the case of Stokes edge wave. For the higher modes we find from the properties of the Laguerre polynomial (26) that the amplitudes of the vorticity and divergence at \( x = 0 \) are respectively
The asymptotic expressions for $\tilde{\zeta}_{jn}(k)$ and $\overline{D}_{jn}(k)$ are given in Tables IV and V respectively. It is evident that the $j = 3$ mode is quite different from the other two; if we allow $f \to 0$, then the vorticity of the $j = 1, 2$ modes vanishes while that of the $j = 3$ mode increases without bound. However, it should be recalled that a unit amplitude of $H_{jn}$ is presumed. Thus a finite vorticity of the $j = 3$ mode can exist for $f = 0$ if there is no surface distortion; this is the vortex flow referred to earlier. Another point worthy of note is that the vorticity of the $j = 3$ mode is virtually independent of the bottom slope $s$.

**TABLE IV. ASYMPTOTIC EXPRESSIONS FOR THE VORTICITY AMPLITUDE AT SHORE $\tilde{\zeta}_{jn}(k)$ FOR $n \gg 1$**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Mode</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k \ll \frac{f^2}{(2n + 1)g_s}$</td>
<td>$n fk$</td>
<td>$\frac{(n + 1)fk}{s}$</td>
<td>$2n(n + 1) \frac{g k^2}{f}$</td>
<td></td>
</tr>
<tr>
<td>$k \gg \frac{f^2}{(2n + 1)g_s}$</td>
<td>$\frac{2n(n + 1) fk}{(2n + 1) s}$</td>
<td>$\frac{2n(n + 1) fk}{(2n + 1) s}$</td>
<td>$\frac{(2n + 1)^2 g k^2}{2f}$</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE V. ASYMPTOTIC EXPRESSIONS FOR THE DIVERGENCE AMPLITUDE AT SHORE $\overline{D}_{jn}(k)$ FOR $n \gg 1$**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Mode</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k \ll \frac{f^2}{(2n + 1)g_s}$</td>
<td>$\frac{n fk}{s}$</td>
<td>$\frac{(n + 1)fk}{s}$</td>
<td>$2n(n + 1) \frac{sg k^3}{f^3}$</td>
<td></td>
</tr>
<tr>
<td>$k \gg \frac{f^2}{(2n + 1)g_s}$</td>
<td>$\frac{2n(n + 1)}{\sqrt{2n + 1}} \frac{\sqrt{g k^3}}{s}$</td>
<td>$\frac{2n(n + 1)}{\sqrt{2n + 1}} \frac{\sqrt{g k^3}}{s}$</td>
<td>$\frac{(2n + 1) g k^2}{2f}$</td>
<td></td>
</tr>
</tbody>
</table>

**THE FORCED WAVE PROBLEM**

1. **Formal Solution of the Forced and/or Initial Value Problem.**

The system of equations (15) to (19), which define the eigenfunctions of the problem, also require an orthogonality condition among the eigenfunctions in the generalized Hilbert sense. This condition is
\[
\int_0^\infty \left\{ \frac{1}{g s x} \left[ X^*_{j_n}(k x) X_{p m}(k x) + Y^*_{j_n}(k x) Y_{p m}(k x) \right] \right. \\
+ Z^*_n(k x) Z_m(k x) \right\} dx = \begin{cases} 
0 & \text{if } p \neq j \text{ or } m \neq n \\
N_{j_n}(k) & \text{if } p = j \text{ and } m = n
\end{cases}
\]

where the asterisk denotes the complex conjugate (for proof, see Appendix A). The quantity \( N_{j_n}(k) \) is the norm of the set of eigenfunctions; it has positive real values for all \( k \) and is closely related to the energy of the normal modes. Using (20) to (22) together with (29) and (30) it can be shown (but not easily) that

\[
N_{j_n}(k) = \left| \frac{2\omega^2_{j_n} - g s k f}{2k\omega_{j_n}(\omega^2_{j_n} - f^2)} \right|.
\]

In the absence of \( f \), \( N_{j_n}(k) \) reduces to \( 1/|k| \) for all \( j, n \).

The formal solution for \( Q_x, Q_y \) and \( \eta \) of the original problem stipulated by (1) to (6) can be represented by suitable linear combinations of the corresponding normal mode functions \( U_{j_n}, V_{j_n} \) and \( H_{j_n} \), respectively. Now, since the latter functions must have a common complex amplitude for a given \( j, n, k \), it follows that the general solutions, incorporating the complete set of edge wave modes for all possible \( k \), must be of the form

\[
Q_x(x, y, t) = \sum_{j=1}^{3} \sum_{n=0}^\infty \int_{-\infty}^{\infty} A_{j_n}(k, t) U_{j_n}(x, y, t; k) \, dk,
\]

\[
Q_y(x, y, t) = \sum_{j=1}^{3} \sum_{n=0}^\infty \int_{-\infty}^{\infty} A_{j_n}(k, t) V_{j_n}(x, y, t; k) \, dk,
\]

\[
\eta(x, y, t) = \sum_{j=1}^{3} \sum_{n=0}^\infty \int_{-\infty}^{\infty} A_{j_n}(k, t) H_{j_n}(x, y, t; k) \, dk,
\]

where the set of (complex) functions \( A_{j_n}(k, t) \) represent collectively the amplitude spectrum of the disturbance (with respect to \( j, n, k \)).

In view of the orthogonality condition (63), together with (1) to (11), it can be shown that

\[
A_{j_n}(k, t) = a_{j_n}(k) + \int_{0}^{t} M_{j_n}(k, t') \, dt',
\]

where

\[
a_{j_n}(k) = \frac{1}{2\pi N_{j_n}(k)} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} \left\{ \frac{1}{g s x} \left[ X^*_{j_n}(k x) Q_x(x, y, 0) \right. \\
+ Y^*_{j_n}(k x) Q_y(x, y, 0) \right\] + Z^*_{n}(k x) \eta(x, y, 0) \right\} e^{-ikx} dx
\]
and
\[ M_{jn}(k, t) = \frac{1}{2\pi N_{jn}(k)} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} \left\{ \frac{1}{g_{sx}} [U_{jn}(x, y, t; k)F_{x}(x, y, t) + V_{jn}(x, y, t; k)F_{y}(x, y, t)] \right\} dx. \] (70)

The derivation of these last three relations is outlined in Appendix B. The function \( a_{jn}(k) \) represents that part of the amplitude spectrum governed by the initial conditions of flow and water level; \( M_{jn}(k, t) \) on the other hand is determined by the forcing functions alone and will be referred to hereafter as the excitation. Note that the excitation represents the time rate of change of the amplitude spectrum.

If we relax condition (5) by allowing a nonuniform line source \( Q_{z}(0, y, t) \) at the water's edge (representing a localized land drainage), then we must add a third function, \( B_{jn}(k, t) \), on the right-hand side of (68). It is shown in Appendix B that the contribution to the amplitude spectrum by the volume source is
\[ B_{jn}(k, t) = \frac{1}{2\pi N_{jn}(k)} \int_{0}^{t} dt' \int_{-\infty}^{\infty} H^{*}(0, y, t'; k)Q_{z}(0, y, t') dy. \] (71)

The effect of localized land drainage is probably of secondary importance, but it may be significant near river mouths during times of exceptional flood stage.

It may be remarked that, in view of the behaviour of the functions \( X_{jn}, Y_{jn}, Z_{n} \) and \( \omega_{jn} \) for negative \( k \), the real part of \( A_{jn} \) is even in \( k \) while the imaginary part is odd, thus assuring real values of the outputs of operations (65) to (67). The imaginary part vanishes identically in the integration over all \( k \). Thus we need consider only the real parts of the integrands of (65) to (67); furthermore, if we restrict \( k \) to positive values only, then the integration over all \( k \) can be replaced by twice the integral over all positive \( k \).

2. Energy Spectrum. The total kinetic and potential energy of the disturbance at any instant is defined by
\[ E(t) = \frac{1}{2} \rho \int_{-\infty}^{\infty} dy \int_{0}^{\infty} \left[ \frac{Q_{x}^{2}}{h} + \frac{Q_{y}^{2}}{h} + g\eta^{2} \right] dx. \] (72)

This can be transformed to a convenient form involving \( A_{jn}(k, t) \) as follows: multiply (65), (66) and (67) by \( Q_{x}/h, Q_{y}/h \) and \( g\eta \), respectively; add the resulting equations and substitute in the integrand of (72); then change the order of integration and summation and apply the
complex conjugate counterpart of (101) as given in Appendix B; this yields

\[
E(t) = 2\pi \rho g \sum_{j=1}^{3} \sum_{n=0}^{\infty} \int_{0}^{\infty} N_{jn}(k)A_{jn}(k, t)A_{jn}^{*}(k, t) \, dk,
\]

which is an analogue of Parseval's theorem. Except for the factor \(2\pi \rho g\), the spatial energy spectrum is given by

\[
S_{jn}(k, t) = N_{jn}(k)|A_{jn}(k, t)|^2.
\]

3. **Initially Static Mound of Water.** If we are interested in the dispersion of free edge waves from an initially static mound of water, then the amplitude spectrum is simply \(a_{jn}(k)\), which is invariant in time, and from (69) we find

\[
a_{jn}(k) = p_{jn}(k) \frac{1}{\pi} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} Z_{n}(kx)\eta(x, y, 0)e^{-iky} \, dx|k|,
\]

where

\[
p_{jn}(k) = \frac{1}{2|k|N_{jn}(k)}.
\]

The latter function has the convenient property that

\[
\sum_{j=1}^{3} p_{jn}(k) = 1.
\]

The spectral distribution in respect to \(n\) and \(k\) is determined largely by the form of \(\eta(x, y, 0)\). However, the partitioning of the energy spectrum among the three discrete modes \((j = 1, 2, 3)\) is determined entirely by the weight function \(p_{jn}(k)\) for an initially static mound of water, since

\[
S_{jn} = p_{jn} \sum_{j=1}^{3} S_{jn}
\]

in this case. Graphs of the weight functions \(p_{jn}(k)\) for some selected values of \(n\) are shown in Fig. 5. Modes 1 and 2 are shown for the fundamental \((p_{3,0} being identically zero), modes 1 and 3 for the higher order waves. The asymptotic expressions for \(p_{jn}(k)\) are listed in Table VI. For the particular case of the fundamental mode

\[
p_{j,0}(k) = \frac{1}{2} \left[ 1 \mp \frac{1}{\sqrt{1 + 4gsk/f^2}} \right],
\]

for \(j = 1, 2\) respectively.
Inspection of Fig. 5 reveals that $p_{1,n} < \frac{1}{2}$ for finite $k$, hence we must conclude that rotation of the earth favors dispersal of energy of an initially static deformation towards the negative $y$ direction (to the south along the east coast of the United States). This effect is most pronounced of course for the longest waves; for the shorter waves the energy is propagated nearly equally in both directions. The quasi-geostrophic mode can contribute about 10% or more to the spectral energy for a given $n$ and $k$ if $(2n + 1)gsk < 9f^2$. For $n = 1, s = 10^{-3}$, this would require an initial disturbance with an effective scale of the order of 2000 km or greater.

**TABLE VI. ASYMPTOTIC EXPRESSIONS FOR THE WEIGHT FUNCTIONS $p_{j,n}(k)$**

<table>
<thead>
<tr>
<th>Condition</th>
<th>Order</th>
<th>$j = 1$</th>
<th>$j = 2$</th>
<th>$j = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k \ll \frac{f^2}{(2n + 1)gsk}$</td>
<td>$n = 0$</td>
<td>$\frac{gsk}{f^2}$</td>
<td>$1 - \frac{gsk}{f^2}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$n &gt; 1$</td>
<td>$(n + 1)\frac{gsk}{f^2}$</td>
<td>$n\frac{gsk}{f^2}$</td>
<td>$1 - (2n + 1)\frac{gsk}{f^2}$</td>
</tr>
<tr>
<td>$k \gg \frac{f^2}{(2n + 1)gsk}$</td>
<td>$n = 0$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$n &gt; 1$</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>$\frac{f^2}{(2n + 1)gsk}$</td>
</tr>
</tbody>
</table>
4. Some General Considerations of Forced Edge Waves. If we suppose that the force field is nil for \( t \leq 0 \) and that the water is initially undisturbed, then, except for the possibility of a volume source at shore (which shall be ignored), the amplitude spectrum is determined entirely by the space-time distribution of the forcing functions for a given bottom slope. As stated at the outset, we will regard the force field for \( t \geq 0 \) to be localized such that \( F_x \) and \( F_y \) vanish at infinity.

The vector-forcing function \( \mathbf{F} \) (of which \( F_x \) and \( F_y \) are components) is determined by the surface wind stress, \( \tau \), and the atmospheric pressure, \( P \), by the relation

\[
\mathbf{F} = \frac{1}{\rho} \left[ -h \nabla P + \tau \right],
\]

where \( \nabla \) is the two-dimensional gradient operator in the \( x, y \)-plane. Using (70), the excitation can consequently be separated into two parts, \( M_{jn}^{(p)}[k, t] \) and \( M_{jn}^{(r)}[k, t] \), related to pressure and stress respectively. Thus

\[
M_{jn}[k, t] = M_{jn}^{(p)}[k, t] + M_{jn}^{(r)}[k, t],
\]

where

\[
M_{jn}^{(p)}[k, t] = \frac{1}{2\pi\rho g N_{jn}(k)} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \left[ U_{jn}^* \frac{\partial P}{\partial x} + V_{jn}^* \frac{\partial P}{\partial y} \right] dx \tag{81}
\]

and

\[
M_{jn}^{(r)}[k, t] = \frac{1}{2\pi\rho N_{jn}(k)} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \left[ \frac{U_{jn}^* \tau_x + V_{jn}^* \tau_y}{gs} \right] dx. \tag{82}
\]

In the above relations the functional dependence notation is dropped for the sake of brevity.

Relation (81) can be transformed to a more convenient form if we employ integration by parts, bearing in mind that \( U_{jn}^* \) vanishes at \( x = 0 \) and that the departure of pressure from normal, \( P' \), approaches zero as \( x \) and \( |y| \to \infty \). Furthermore, if we make use of the conjugate counterparts of (9) and (45), we find that

\[
M_{jn}^{(p)}[k, t] = \frac{ikp_{jn}\omega_n}{\pi \rho g} \int_{-\infty}^{\infty} dy \int_{0}^{\infty} H_{jn}^*(x, y, t; k) P'(x, y, t) dx. \tag{83}
\]

The excitation of edge waves by atmospheric pressure has been dealt with at some length by Greenspan (1956). His analysis for a storm moving parallel to the coast indicates that, for a Gaussian distribution of \( P' \), it is primarily the fundamental mode which is excited (at least for distributions in which the horizontal scale is comparable
to that of a typical hurricane). Furthermore, the greatest effect occurs when the path of the low pressure center is close to the coast. These results, insofar as \( P' \) alone is concerned, should apply equally well for the inertiogravitational edge waves considered here, since the integral involved in (83) is independent of \( f \). It need only be remarked in passing that the partitioning of the energy of the fundamental waves to modes \( j = 1 \) and 2 is governed, at least in part, by \( p_{j,0}(k) \). The implications of this are explored in some detail by Kajiura (1958).

In the remaining discussion we turn our attention to the question of excitation of edge waves by wind stress. There are two different transformations of (82) which will be considered. The first involves the divergence and curl of the wind stress and the second involves the divergence and vorticity of the normal mode flow.

If we employ the conjugate counterparts of (7) and (8) and keep in mind the periodic character of the normal modes, we find that

\[
U_{jn}^{*} = \frac{g s x}{(\omega_{jn}^2 - f^2)} \left[ -i \omega_{jn} \frac{\partial H_{jn}^{*}}{\partial x} + f \frac{\partial H_{jn}^{*}}{\partial y} \right] \tag{84}
\]

and

\[
V_{jn}^{*} = \frac{g s x}{(\omega_{jn}^2 - f^2)} \left[ -i \omega_{jn} \frac{\partial H_{jn}^{*}}{\partial y} - f \frac{\partial H_{jn}^{*}}{\partial x} \right]. \tag{85}
\]

Substituting (84) and (85) in relation (82) and integrating by parts, using the condition that \( \tau \) vanishes as \( x \) and \( |y| \rightarrow \infty \) for a localized disturbance, we obtain the first transformation of the stress excitation function, namely

\[
M_{jn}^{(\tau)}[k, t] = -i R_{jn} \omega_{jn} \left[ - \int_{-\infty}^{\infty} dy \int_{0}^{\infty} H_{jn}^{*} \text{Div} \ \tau \ dx \right.
+ \left. \int_{-\infty}^{\infty} (H_{jn}^{*} \tau_x)_{x=0} dy \right] + R_{jn} f \left[ \int_{-\infty}^{\infty} dy \int_{0}^{\infty} H_{jn}^{*} \text{Curl} \ \tau \ dx \right.
- \left. \int_{-\infty}^{\infty} (H_{jn}^{*} \tau_y)_{z=0} dy \right], \tag{86}
\]

where the following abbreviations are employed for convenience:

\[
\text{Div} \ \tau = \frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}, \tag{87}
\]

\[
\text{Curl} \ \tau = \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}. \tag{88}
\]
and

$$R_{jn} = \frac{k p_{jn}(k)}{\pi \rho (\omega_{jn}^2 - f^2)}. \quad (89)$$

For an axially symmetric atmospheric disturbance, the divergence and curl of the wind stress are readily determined from distributions of the radial and rotary components of the stress respectively.

The first two integrals in (86) represent the effect of "piling up" of water by the action of a converging wind pattern and by the onshore wind component at the coast respectively. The last two integrals represent the quasigeostrophic response of the water level to the circulation or longshore flow caused by a rotary wind pattern and by the longshore wind component at the coast respectively. In the absence of the earth's rotation, the wind curl and longshore wind component at the coast would have no effect on the water level. The relative partitioning of excitation among different edge wave modes is governed by $R_{jn} \omega_{jn}$ for the effect of convergent winds or onshore wind component and by $R_{jn}$ for the case of rotary winds or longshore wind component.

A second transformation of (82) is obtained by expressing the stress components in the form

$$\tau_x = - \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial y}, \quad (90)$$

$$\tau_y = - \frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial x}, \quad (91)$$

where the potential functions $\Phi(x, y, t)$ and $\Psi(x, y, t)$ satisfy the relations

$$\nabla^2 \Phi = - \text{Div } \tau, \quad (92)$$

$$\nabla^2 \Psi = \text{Curl } \tau. \quad (93)$$

In order to define $\Phi$ and $\Psi$ uniquely we may specify that $\Phi$ and $\Psi$, together with their gradients, vanish at infinite $x$ and $y$. The interpretation of the above potential functions is particularly simple for an axially symmetric stress field; in this case $\Phi$ and $\Psi$ are determined respectively by the integrals of the radial and rotary components of stress with respect to distance from the axis of symmetry. Substituting (90) and (91) in (82), integrating by parts and making use of the conjugate counterparts of (53) and (54), yields
\[ M_{jn}(r)[k, t] = \frac{k p_{jn}}{\pi \rho g} \left\{ \int_{-\infty}^{\infty} dy \int_{0}^{\infty} (\Phi D_{jn}^* - \Psi \xi_{jn}^*) dx \right. \\
+ \frac{\omega_{jn}}{s} \left[ -i \int_{-\infty}^{\infty} (\Phi H_{jn})_{z=0} dy \right. \\
+ \left( \frac{2n + 1}{2n + 1} \omega_{jn} - f \int_{-\infty}^{\infty} (\Psi H_{jn})_{z=0} dy \right) \right\}. \tag{94} \]

Since \( D_{jn} \) and \( \xi_{jn} \) vanish identically for \( n = 0 \), we have as a special case of (94)

\[ M_{jo}(r)[k, t] = -\frac{k p_{jo} \omega_{jo}}{\pi \rho g s} \int_{-\infty}^{\infty} [(\Psi + i\Phi)H_{jo}]_{z=0} dy. \tag{95} \]

This relation implies a rather surprising result which is of considerable interest from a practical standpoint: The excitation of the fundamental edge wave mode by wind stress is determined uniquely by variation of the potentials \( \Phi \) and \( \Psi \) along the shore alone. For an axially symmetric wind stress vortex, the functions \( \Phi \) and \( \Psi \) can be related to the pressure anomaly, \( P' \), by virtue of the gradient wind equation and by a knowledge of the direction of the surface stress relative to the isobars. Consequently, in such a model the distribution of pressure along shore determines the stress-induced fundamental edge waves. This immediately leads to the following corollary: If the atmospheric disturbance is confined to a finite region offshore, such that \( \Phi \) and \( \Psi \) are zero along shore, then the wind stress field associated with the disturbance will not produce a fundamental edge wave. However, the fundamental may be excited to some degree by the direct effect of pressure. Evidently the maximum excitation of fundamental edge waves by wind stress occurs when the center of the atmospheric vortex is close to shore. It may also be noted, on comparing (83) and (95), that the ratio of the excitation by wind stress to that by pressure (for the fundamental waves) is inversely proportional to the bottom slope, as we might have anticipated.

Relation (94) is useful in ascertaining the relative partitioning of excitation between the gravitational edge wave modes \((j = 1, 2)\) and the quasigeostrophic mode \((j = 3)\) for the higher order edge waves \((n \geq 1)\). For a convergent wind stress system, the products \( p_{jn} D_{jn}^* \) and \( p_{jn} \omega_{jn} \) determine the partitioning of \( M_{jn}(r) \) among the three separate modes. Inspection of the asymptotic expressions for \( \omega_{jn}, D_{jn} \) and \( p_{jn} \) (Tables II, V and VI) reveals that the gravity modes are favored for large \( k \); however, a nearly equal partitioning of \( M_{jn}(r) \) between the gravity modes and the quasigeostrophic mode occurs for small \( k \).
For a *rotary* wind stress system, the products $p_{jn} r_{jn}^*$ and

$$\left[\frac{(2n + 1)f - \omega_{jn}}{(2n + 1)\omega_{jn} - f}\right] p_{jn}$$

determine the relative partitioning of $M_{jn}^{(r)}$ among the three modes. Using (30) and Tables II, IV and VI it is found that the quasigeostrophic mode is favored for all $k$. In a typical hurricane the rotary component of wind stress is usually of greater magnitude than the radial (or convergent) component. We should therefore expect that, if higher order edge waves are generated by a hurricane, the quasigeostrophic mode will play a significant role.

The relative importance of the higher order edge waves, compared with the fundamental, depends to a large extent on the actual distribution of stress in the hurricane. However, a simple special case may serve to demonstrate that the higher order waves are not negligible. In order to examine this question we will return to relation (86). For simplicity we will suppose $f = 0$, as in Greenspan’s analysis of pressure-induced edge waves; furthermore, we will consider that the field of stress is divergenceless, as would be the case for an axially symmetric system with purely rotary stresses. In this special case, (86) reduces to

$$M_{jn}^{(r)}[k, t] = \frac{i}{2\pi \rho} \sqrt{\frac{k}{(2n + 1)gs}} e^{-\omega_{jn}t} \int_{-\infty}^{\infty} \tau_x(0, y, t) e^{-iky} dy \quad (96)$$

for $j = 1, 2$ respectively. The absolute value of the excitation decreases rather slowly with increasing $n$, and it is evident that the higher order waves play a significant role in this special case.

The above example, in fact, raises the question of the convergence of the formal solution given by relations (65) to (67). Examination of the basic relations (1) and (2) at $x = 0$ reveals that the solution may indeed be questionable when $\tau$ differs from zero at the water’s edge. The difficulty is not merely a mathematical one; undoubtedly it stems from the omission of bottom stress in the formulation of the problem. This matter definitely needs further attention; it may well be that we must impose the condition that the net stress at the water’s edge should vanish if we are to get sensible results from the linear wave theory. In any event we cannot conclude, *a priori*, that the introduction of bottom stress will completely suppress the higher order edge waves for a given wind stress distribution, for in the more general theory the large divergence of net stress which is required near shore should play a role similar to that of $\tau_x(0, y, t)$ in the special case above.
SUMMARY AND CONCLUSIONS

It has been attempted herein to determine some of the important characteristics of edge waves on a rotating flat earth and to discuss the question of excitation of different edge wave modes from a general point of view. The linearized mathematical development is confined to the case of very small bottom slope and employs the hydrostatic pressure approximation. The vertical component of the earth's vorticity \((f)\) is regarded as a constant. Furthermore, bottom stress is neglected and consequently no allowance is made for the dissipation of energy in the system. In the discussion of forced disturbances it is considered that the forcing functions are localized and consequently our attention is directed exclusively to the question of excitation of edge waves.

The periodic edge wave modes have a surface configuration which is characterized in the offshore direction by Laguerre functions of integer order \(n\), just as in the case of no Coriolis force. However, for a given order \(n(\geq 1)\) and a given longshore wave length, there are three different frequencies (hence phase speeds) which are possible. Two of these modes are the counterpart of pure gravity edge waves in a nonrotating co-ordinate system; the other mode corresponds to a quasi-geostrophic wave with very small speed. In the case of the fundamental edge waves \((n = 0)\), only the first two modes are possible. These inertiogravitational edge waves are the counterpart of Stokes edge wave; however, the phase speed is dependent upon the direction of propagation along the coast. The important characteristics of the various edge wave modes and some general comments in regard to the excitation of these waves are summarized below.

A. General Comments on Inertiogravitational Edge Wave Modes \((j = 1, 2; \text{all } n)\)

(1) For a given longshore wave length, the period of free waves progressing to the right is less than that of waves progressing to the left (facing shore from the sea in the northern hemisphere).

(2) For a given longshore wave length, the phase speed of waves progressing to the right is greater than that of waves progressing to the left.

(3) For sufficiently small longshore wave length (less than 10 km at midlatitudes for \(s = 10^{-3}\)), the effect of the earth's rotation is negligible and the properties of waves progressing in either direction along shore are essentially the same.
B. Fundamental Inertiogravitational Edge Waves \((j = 1, 2; n = 0)\)

1. The magnitude of the group velocity of these waves is independent of the direction of propagation along shore.
2. The limiting value of the group velocity is \(gs/f\); this is approached for extremely great longshore wave lengths.
3. The fundamental waves are horizontally divergenceless and irrotational (in the sense that the vertical component of vorticity relative to the earth is zero); this is one of the few characteristics which is independent of the earth's rotation.
4. For sufficiently large wave length, the waves which propagate to the left are associated with quasigeostrophic flow; this is quite different from the character of the fundamental long waves which move to the right.

C. Inertiogravitational Edge Waves of Higher Order \((j = 1, 2; n \geq 1)\)

1. The magnitude of the maximum group velocity for waves moving to the right and left is \((n + 1)gs/f\) and \(ngs/f\) respectively.
2. The higher order inertiogravitational waves possess horizontal divergence; for relatively small longshore wave lengths the divergence is independent of the earth's rotation but is strongly dependent upon bottom slope.
3. The higher order inertiogravitational waves also possess vorticity relative to the earth; the magnitude of the vorticity is proportional to the vertical component of the earth's vorticity \(f\) and is also strongly dependent upon bottom slope.

D. Quasigeostrophic Edge Waves \((j = 3; n \geq 1)\)

1. These waves are of very small frequency or large period (greater than 12 pendulum hours) and move only to the left along shore (for example, to the south along the east coast of the United States).
2. The magnitude of the maximum group velocity (and phase speed) is the same as that of fundamental edge waves \((gs/f)\); however, for longshore wave lengths less than about 3000 km \((s = 10^{-3})\), the phase speed is very small and the group velocity virtually zero.
3. The small group velocity \((\lambda > 3000 \text{ km})\) implies that these waves are not effective in the dispersion of energy.
4. The oscillatory longshore flow is in quasigeostrophic balance with the oscillatory offshore slope of the sea surface for all longshore wave lengths and for all \(n \geq 1\).
5. The quasigeostrophic edge waves possess large vorticity relative to the earth; the vorticity is independent of bottom slope and inversely proportional to \(f\).
E. Forced or Initial Disturbances

The general considerations of the dispersion of energy from an initially static mound of water disclose that rotation of the earth favors a greater propagation of energy to the left than to the right along shore. Some energy will remain temporarily in the region of the initial disturbance in the form of a quasistationary disturbance with quasigeostrophically balanced flow if the initial disturbance is of sufficiently large horizontal scale (of 2000 km or greater). A formula for the total energy of a general edge wave disturbance is given in terms of the complex amplitude spectrum of the disturbance.

In respect to forced edge waves due to a localized atmospheric disturbance, we may note the following general findings:

1. Greenspan's conclusion that the higher order edge waves are not effectively excited by a Gaussian pressure distribution should apply equally well in the presence of the earth's rotation; however, the response of the fundamental waves differs because of the dependency of these waves on the direction of propagation.

2. Excitation of fundamental edge waves by wind stress can be determined from a knowledge of two stress potentials along shore; these potentials are determined from the convergence and curl of the wind stress field.

3. The ratio of excitation of fundamental waves by wind stress to that by pressure is inversely proportional to the bottom slope.

4. The present analysis indicates that the higher order inertio-gravitational waves can be excited significantly by the onshore component of stress at the shore or by the convergent part of the wind field, although this is not conclusive in view of items (6) and (7) below.

5. The quasigeostrophic mode can be excited by the longshore component of the wind at the shore or by the curl of the wind stress field.

6. The convergence of the series solutions for a stress-induced disturbance is questionable if a finite net stress is allowed at the shore line.

7. The omission of bottom stress apparently imposes serious limitations on the applicability of the present theory insofar as stress-induced edge waves are concerned. The need for further investigation is evident.

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APPENDIX A

In order to establish the orthogonality condition (63) we consider any two sets of eigenfunctions \( \{X_{jn}(kx), Y_{jn}(kx), Z_{n}(kx)\} \) and \( \{X_{pm}(kx), Y_{pm}(kx), Z_{m}(kx)\} \) which, together with their associated characteristic frequencies \( \omega_{jn}(k) \) and \( \omega_{pm}(k) \), respectively, satisfy the basic relations (15) to (17) and conditions (18), (19). The integers \( p(1, 2, 3) \) and \( m(0, 1, 2, \ldots) \) are considered independent of \( j \) and \( n \), but the two sets of functions have a common \( k \). The complex conjugate functions \( \{X_{jn}^*, Y_{jn}^*, Z_{n}^*\} \) satisfy the conjugate relations formed from (15) to (17), using the general rule that the conjugate of a product equals the product of the conjugates of the individual factors. This implies that the equations which the conjugate eigenfunctions satisfy are the same as (15) to (17) except that \( i \) is replaced by \( -i \) and \( \omega \) is taken as \( \omega_{jn} \). We will refer to these equations as (15c) to (17c).

The conjugate relations (15c) to (17c) formed from \( \{X_{jn}^*, Y_{jn}^*, Z_{n}^*; \omega_{jn}\} \) are multiplied by \( X_{pm}/gsx, Y_{pm}/gsx \) and \( Z_{m} \), respectively. Similarly, the relations (15) to (17) formed from \( \{X_{pm}, Y_{pm}, Z_{m}; \omega_{pm}\} \) are multiplied by \( X_{jn}^*/gsx, Y_{jn}^*/gsx \) and \( Z_{n}^* \), respectively. If we add the six resulting equations, we find that the terms involving \( f \) and \( k \) drop out and the final quadratic relation is

\[
i(\omega_{pm} - \omega_{jn}) \left\{ \frac{1}{gsx} [X_{pm}X_{jn}^* + Y_{pm}Y_{jn}^*] + Z_{m}Z_{n}^* \right\} + \frac{d}{dx} [X_{jn}^*Z_{m} + X_{pm}Z_{n}^*] = 0. \tag{97}\]

The above relation is the normal mode counterpart of an energy equation. If we integrate with respect to \( x \) from 0 to \( \infty \) and bear in mind conditions (18) and (19), then it follows that

\[
(\omega_{pm} - \omega_{jn}) \int_{0}^{\infty} \left\{ \frac{1}{gsx} [X_{pm}X_{jn}^* + Y_{pm}Y_{jn}^*] + Z_{m}Z_{n}^* \right\} dx = 0. \tag{98}\]

Thus if \( \omega_{pm} \neq \omega_{jn} \), then the integral must vanish and the orthogonality condition (63) follows at once. On the other hand, if \( \omega_{pm} = \omega_{jn} \), which requires that the two sets of eigenfunctions are identical, then the integral in (98) does not vanish since the integrand represents the sum of squares of real numbers. The integral in this case is the norm of the eigenfunctions in a collective sense and is given by

\[
N_{jn}(k) = \int_{0}^{\infty} \left\{ \frac{1}{gsx} [X_{jn}^2 + Y_{jn}^2] + Z_{n}^2 \right\} dx, \tag{99}\]

which is a positive real number for any \( j, n \) and \( k \).
APPENDIX B

An outline of the derivation of relations (68) to (71) is given herein. The form of the solutions for $Q_x$, $Q_y$ and $\eta$ are given by (65), (66) and (67), respectively, where the complex amplitude-phase functions $A_{jn}(k, t)$ are to be determined. The latter set of functions represents a particular set of integral transforms of the functions $Q_x$, $Q_y$ and $\eta$ in a collective sense; the first step is to find out the form of this transform. The next step is to determine the differential equation in $t$ which the transforms $A_{jn}(k, t)$ must satisfy.

The first step is carried out as follows: multiply (65), (66) and (67) by $X_{pm}/gsx$, $Y_{pm}/gsx$ and $Z_m^*$, respectively; introduce relations (43), (44) and (45); add the resulting three equations; then integrate with respect to $x$ from 0 to $\infty$. In view of the orthogonality condition (63), the resulting equation reduces to

$$\int_{-\infty}^\infty [N_{jn}(k)A_{jn}(k, t)e^{\omega_{jn}(k)t}]e^{ikx} \, dk = \int_0^\infty \left[ \frac{X_{jn}^* Q_x + Y_{jn}^* Q_y}{gsx} + Z_n^* \eta \right] \, dx,$$

(100)
since the only term in the double summation which does not vanish is that for which $p = j$ and $m = n$. We can solve formally for $A_{jn}$ by taking the Fourier transform of (100), and if we employ the conjugate counterparts of (43), (44) and (45) we find

$$A_{jn}(k, t) = \frac{1}{2\pi N_{jn}(k)} \int_{-\infty}^\infty dy \int_0^\infty \left[ \frac{U_{jn}^* Q_x + V_{jn}^* Q_y}{gsx} \right. + H_{jn}^* \eta \left. \right] \, dx.$$

(101)

This is the particular Fourier-Laguerre transform appropriate to the present problem.

The next step is to reduce the basic relations (1), (2), (3) and the normal mode equations (7), (8), (9) to a form involving the above transform as well as an analogous transform of the forcing functions. Note first that, since the set of functions $\{U_{jn}, V_{jn}, H_{jn}\}$ satisfies relations (7) to (9), the conjugate set $\{U_{jn}^*, V_{jn}^*, H_{jn}^*\}$ must also satisfy these relations. The procedure is then as follows: multiply the relations (7), (8) and (9) formed from the set $\{U_{jn}^*, V_{jn}^*, H_{jn}^*\}$ by $Q_x/gsx$, $Q_y/gsx$ and $\eta$, respectively; multiply (1), (2) and (3) by $U_{jn}^*/gsx$, $V_{jn}^*/gsx$ and $H_{jn}^*$, respectively; then add the six resulting equations. Upon collecting terms we find
\[
\frac{\partial}{\partial t} \left[ \frac{U_{jn} * Q_z + V_{jn} * Q_v}{gsx} + H_{jn} * \eta \right] + \frac{\partial}{\partial x} \left[ U_{jn} * \eta + H_{jn} * Q_z \right] + \frac{\partial}{\partial y} \left[ V_{jn} * \eta + Q_v H_{jn} * \right] = \frac{U_{jn} * F_z + V_{jn} * F_v}{gsx}.
\]

(102)

If we integrate with respect to \( x \) and \( y \) over the entire semi-infinite half plane \( (x \geq 0) \) and make use of (101) and condition (6) we find

\[
2 \pi N_{jn}(k) \frac{d}{dt} [A_{jn}(k, t)] = \int_{-\infty}^{\infty} (U_{jn} * \eta + Q_z H_{jn} *)_{x=0} dy
\]

\[
+ \int_{-\infty}^{\infty} dy \int_{0}^{\infty} \left[ \frac{U_{jn} * F_z + V_{jn} * F_v}{gsx} \right] dx.
\]

(103)

Conditions (5) and (10) imply that the first integral on the right side of (103) vanishes. However, if we relax the condition (5) and allow a finite volume source at shore, then the term \( Q_z H_{jn} * \) does not vanish at \( x = 0 \). It therefore follows immediately from (103) that

\[
A_{jn}(k, t) = A_{jn}(k, 0) + B_{jn}(k, t) + \int_{0}^{t} M_{jn}(k, t) dt,
\]

(104)

where \( B_{jn}(k, t) \) and \( M_{jn}(k, t) \) are as defined by (70) and (71). The term \( B_{jn}(k, t) \) of course vanishes in the absence of a volume source at shore. To complete the solution we note that \( A_{jn}(k, 0) \) can be evaluated by direct application of (101) since \( Q_z, Q_v \) and \( \eta \) are presumed to be known at \( t = 0 \). If we make use of (43) to (45) we find that \( A_{jn}(k, 0) = a_{jn}(k) \), as defined by (69).