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The fundamental notion of statistical mean values in fluid mechanics was first introduced by Reynolds (1894). In his description of the turbulent motion he started from the molecular disorder. By considering the mean value and correlations in the molecular motion, as it was shown before, one obtains the equation of Stokes-Navier (1845, 1822). The solutions of these equations were designated by Reynolds as "mean motion." The second step in Reynolds' development is the consideration of the turbulent fluctuation. He assumes, for instance, that the apparently parallel flow in a cylindrical tube consists of a fluctuating motion characterized by the fact that the motion, at every instant, satisfies the Stokes-Navier equation. The parallel motion obtained by the formation of average values is called the "mean mean motion."

Reynolds' most important contributions were the definition of the mean values for the so-called Reynolds stresses and the recognition of the analogy between momentum transfer, transfer of heat and matter in the turbulent motion.

In the decades following Reynolds' discoveries, turbulence theory was directed to find semiempirical laws for the mean mean motion by methods borrowed from the kinetic theory of gases, i.e., from the theory of the mean motion.

Prandtl's (1925, 1927) ideas on momentum transfer and Taylor's (1915, 1932) suggestions concerning vorticity transfer belong to the most important contributions of this period. But I believe that my formulation of the problem (1930) by the application of the similarity principle has the merit of being more general and independent of the methods of the kinetic theory of gases. This theory of mine led to the discovery of the logarithmic law of velocity distribution for the case of homologous turbulence. I call the turbulence homologous if the

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distribution of turbulence fluctuations and their correlations are identical at every point of the field. They differ only in scale.

The next important step was the definition of isotropic turbulence by Taylor (1935, 1936). Apparently the case of homologous turbulence, i.e., the shear motion, is too complicated for a fundamental attack by statistical methods. It was Taylor’s fortunate idea to simplify the problem by the consideration of a uniform and isotropic field of turbulent fluctuation. Since such a field can be realized at least approximately in the wind tunnel, the possibility was given for an experimental check of the statistical ideas.

The next period in the development of the theory of turbulence was devoted to the analysis of the quantities which are accessible to measurements in a wind tunnel stream. These quantities are the correlation functions and the spectral function. The general mathematical analysis of the correlations was executed by Howarth and myself (von Kármán and Howarth, 1938). One has to consider five scalar functions \( f(r), g(r), h(r), k(r), l(r) \). These functions determine all double and triple correlations between arbitrary velocity components observed at two points because of the tensorial character of the correlations. The two scalar functions for the double correlations are defined as follows:

\[
\begin{align*}
    f(r) &= \frac{u_1(x_1, x_2, x_3) u_1(x_1 + r, x_2, x_3)}{u_1^2}, \\
    g(r) &= \frac{u_1(x_1, x_2, x_3) u_1(x_1, x_2 + r, x_3)}{u_1^2}.
\end{align*}
\]

Because of the continuity equation for incompressible fluids, \( g = f + \frac{r}{2} \frac{df}{dr} \). For the same reason the triple correlations \( h, k, \) and \( l \) can be expressed by one of them, i.e., by

\[
\begin{align*}
    h(r) &= \frac{[u_1(x_1, x_2, x_3)]^2 u_1(x_1 + r, x_2, x_3)}{[u_1^2]^{3/2}}.
\end{align*}
\]

In addition, we also deduced a differential equation from the Stokes-Navier equation which gives the relation between the time derivative of the function \( f \) and the triple correlation function \( h \).

\[
\begin{align*}
    \frac{\partial}{\partial t} (\overline{u^2f}) + 2[\overline{u^2}]^{3/2} \left( \frac{\partial h}{\partial r} + \frac{4}{r} h \right) &= 2\nu \overline{u^2} \left( \frac{\partial^3 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right)
\end{align*}
\]

We discussed this equation in two special cases:
a) Small Reynolds Number—in this case the triple correlations can be neglected and one obtains a self-preserving form for the double correlation function as function of \( r/\lambda \), where \( \lambda \) is defined by the relation

\[
\frac{d\overline{u^2}}{dt} = -10\nu \frac{\overline{u^2}}{\lambda^2}.
\]  

b) Large Reynolds Number—in this case the terms containing the viscosity can be neglected for not too small values of \( r \), and the functions \( f \) and \( h \) are assumed to be functions of the variable \( r/L \), \( L \) being a length characterizing the scale of turbulence. The hypothesis of self-preserving correlation function leads to the following special results. One can consider three simple cases:

1. \( L = \) constant; then we have \( \overline{u^2} \sim t^{-2} \) (Taylor).

2. Loitsianskii (1939) has shown that if the integral \( \overline{u^2} \int_0^\infty f(r) r'dr \) exists, it must be independent of time and consequently \( \overline{u^2}L^5 = \) constant. Then \( \overline{u^2} \sim t^{-0.17} \), \( L \sim t^{0.17} \).

3. If the self-preserving character is extended to all values of \( r \), i.e., also near \( r = 0 \), one has \( \overline{u^2} \sim t^{-1} \), \( L \sim t^{1/2} \) (Dryden, 1943).

On the other hand, Taylor (1938) introduced a spectral function for the energy passing through a fixed cross section of a turbulent stream as the Fourier transform \( \mathcal{F}_0(n) \) of the correlation function \( f(r) \). The relations between \( \mathcal{F} \) and \( f \) are given by the following equations:

\[
f(r) = \int_0^\infty \mathcal{F}_0(n) \cos \frac{2\pi nr}{U} \, dn,
\]

\[
\mathcal{F}_0(n) = \frac{4u_1^2}{U} \int_0^\infty f(r) \cos \frac{2\pi nr}{U} \, dr.
\]  

In these equations \( n \) is the frequency of the fluctuation of the uniform velocity \( U \) as function of time. Relative to the stream, \( \mathcal{F}_0(n) \) can be replaced by \( \mathcal{F}_1(\kappa_1) \), where \( \kappa_1 = \frac{2\pi n}{U} \), i.e., the wave number of the fluctuation, measured in the \( x_1 \)-direction.

It is seen that in this period of the development of the turbulence theory the analytical and experimental means for the study of isotropic turbulence were clearly defined, but with the exception of the case of very small Reynolds numbers no serious attempt was made to
find the laws for the shapes of either the correlation or the spectral functions. I believe this is the principal aim of the period in which we find ourselves at present. Promising beginnings were made by Kolmogoroff (1941), Onsager (1945), Weizsäcker (1946), and Heisenberg (1946). I do not want to follow the special arguments of these authors. Rather, I want to define the problem clearly and point out the relations between assumptions and results.

First, we will assume that the three components of the velocity in an homogeneous isotropic turbulent field, at any instant, can be developed in the manner of Fourier’s integrals

$$u_i = \int \int \int \int \limits_{-\infty}^{\infty} Z_i(\kappa_1, \kappa_2, \kappa_3, t) e^{i(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)} \, d\kappa_1 d\kappa_2 d\kappa_3. \tag{5}$$

Second, the intensity of the turbulent field shall be characterized by the quadratic mean value $u_i^2$ (level of the turbulence). Also, there exists a function $\mathcal{F}(\kappa)$, such that $[u_i^2]_{i=0} = \int \mathcal{F}(\kappa') \, d\kappa'$, where the symbol $[u_i^2]_{i=0}$ means a partial mean value of the square of the velocity, the averaging process being restricted for such harmonic components whose wave numbers $\kappa_1, \kappa_2, \kappa_3$ satisfy the relation

$$\kappa_1^2 + \kappa_2^2 + \kappa_3^2 \leq \kappa^2.$$

If such a function exists, it is connected with the spectral function of Taylor $\mathcal{F}_1(\kappa_1)$ by the relation

$$\mathcal{F}_1(\kappa_1) = \frac{1}{4} \int \frac{1}{\kappa^3} \mathcal{F}(\kappa) (\kappa^2 - \kappa_1^2) \, d\kappa. \tag{6}$$

This relation was found by Heisenberg (1946). It expresses the geometrical fact that all oblique waves (Fig. 1) with wave length

$$\frac{2\pi}{\kappa} < \frac{2\pi}{\kappa_1}$$

necessarily contribute in the one-dimensional analysis to the waves with wave length

$$\frac{2\pi}{\kappa_1}.$$

Third, it is evident that there must be an equation for the time derivative of $\mathcal{F}(\kappa)$ which corresponds to the equation for the time derivative of $f(r)$ which has been found by Howarth and myself.
The physical meaning of this equation is evident. Let us start from the energy equation

\[ \frac{1}{2} \frac{\partial u_i^2}{\partial t} + \left( u_i u_j + \delta_{ij} \frac{p}{\rho} \right) \frac{\partial u_i}{\partial x_j} = \nu \frac{\partial^2 u_i}{\partial x_i^2} u_j. \] (7)

The right side represents the energy dissipation by viscous forces. The second term on the left side is the work of the Reynolds stresses; it represents a transfer of energy without actual dissipation. Our problem is to find \( \frac{\partial \mathcal{F}}{\partial t} \) by Fourier analysis and averaging process. One finds the contribution of the viscous forces to be equal to \( -2\nu \mathcal{F}(\kappa)\kappa^2 \). Hence we write formally

\[
\frac{\partial \mathcal{F}}{\partial t} + \mathcal{W}_\kappa = -2\nu \kappa^2 \mathcal{F}(\kappa). \quad (8)
\]

Here \( \mathcal{W}_\kappa d\kappa \) is the balance for the energy contained in harmonic com-

Figure 1. Contribution of oblique waves to plane waves in direction of \( x_1 \).
ponents contained in the interval \( d\kappa \); obviously \( \int_0^\infty \mathcal{W} d\kappa = 0 \). C. C.

Lin (personal communication) has shown that \( \mathcal{W}_\kappa = 2\mathcal{K}\kappa^2 \), where \( \mathcal{K}(\kappa) = 2(\kappa^2\mathcal{K}_1''(\kappa) - \kappa\mathcal{K}_1'(\kappa)) \) and

\[
\mathcal{K}_1(\kappa) = \frac{2a^3/2}{\pi} \int_0^\infty h(\tau) \frac{\sin(\kappa\tau)}{\kappa} d\tau.
\]

Unfortunately this relation does not help, as far as the determination of \( f \) and \( h \) is concerned. For example, if one expresses \( h \) in terms of \( f \) by means of the Kármán-Howarth equation, calculates \( \mathcal{W}_\kappa \) and then substitutes the result in equation (8), one obtains an identity. It appears that at the present time one needs some additional physical assumption.

**Fourth**, we assume that \( \mathcal{W}_\kappa \) can be expressed in the form;

\[
\mathcal{W}_\kappa = \int_0^\infty \Theta\{\mathcal{F}(\kappa), \mathcal{F}(\kappa'), \kappa, \kappa'\} d\kappa'.
\]

The physical meaning of this assumption is the existence of a transition function for energy between the intervals \( d\kappa \) and \( d\kappa' \) which depends only on the energy density and the wave numbers of the two intervals. It follows from this definition that by interchanging \( \kappa \) and \( \kappa' \) one has

\[
\Theta\{\mathcal{F}(\kappa), \mathcal{F}(\kappa'), \kappa, \kappa'\} = - \Theta\{\mathcal{F}(\kappa'), \mathcal{F}(\kappa), \kappa', \kappa\}.
\]

It must be noted that our assumption probably cannot be exact. It is very probable that the values of \( \mathcal{F} \) for the difference and the sum of \( \kappa \) and \( \kappa' \) also enter in the transition function. I believe that the assumption gives a fair approximation when \( \kappa \) and \( \kappa' \) are very different, but it is certainly untrue if \( \kappa \) and \( \kappa' \) are nearly equal.

**Fifth**, we furthermore specify the function \( \Theta \) in the following way;

\[
\Theta = - C \mathcal{F}^a(\kappa) \mathcal{F}^{a'}(\kappa') \kappa^\beta k'^{\beta'}; \quad C = \text{const.}
\]

It follows from dimensional reasoning that

\[
\alpha + \alpha' = 3/2, \quad \beta + \beta' = 1/2.
\]

As a result of the sequence of assumptions given above, we obtain the equation;

\[
\frac{\partial \mathcal{F}}{\partial t} = C \left[ \int_0^{\infty} \int_0^{\infty} \mathcal{F}^{3/2-\alpha(\kappa')\kappa_1^{1/2-\beta} d\kappa'} - \mathcal{F}^{3/2-\alpha\kappa_1^{1/2-\beta}} \int_0^{\infty} \mathcal{F}^{a(\kappa')\kappa'^{\beta} d\kappa'} \right] - 2\nu \kappa^2 \mathcal{F}.
\]
Obviously, if \( \mathcal{F} \) is known for \( t = 0 \), equation (12) determines the values of \( \mathcal{F} \) for all times. If one neglects the first term on the left side, which represents the decay of turbulence, and chooses the specific values \( \alpha = 1/2, \beta = -3/2 \), one arrives at the theory proposed by Heisenberg (1946).

Let us consider the case of large Reynolds numbers but assume that \( \kappa \) is not so large that the term containing the viscosity coefficient becomes significant. Let us assume also that the first term on the left side is small by comparison to the second term. Physically, this means that the energy entering in the interval \( d\kappa \) is equal to the energy which leaves the interval. Then one has the relation:

\[
\mathcal{F} \kappa^\beta \int_0^\kappa \mathcal{F}^{3/4-\alpha}(\kappa') \kappa'^{1/2-\beta} d\kappa' = \mathcal{F}^{3/2-\alpha} \kappa^{1/2-\beta} \int_\kappa^{\infty} \mathcal{F}(\kappa') \kappa'^{\beta} d\kappa'.
\]  

(13)

This equation is satisfied by the solution \( \mathcal{F}(\kappa) \sim \kappa^{-5/3} \), as one can easily see by substitution in equation (13). This result is independent, evidently, of the special choice of \( \alpha \) and \( \beta \). That is the reason why it was independently found by Onsager (1945), Kolmogoroff (1941) and Weizsäcker (1946). It is essentially a consequence of dimensional considerations. Let us now stay with the case of large Reynolds numbers by neglecting again the viscosity term while retaining the first term on the left side. In other words, we consider the actual process of decay at large Reynolds numbers. Let us assume that \( \mathcal{F} \) is a function of a nondimensional variable \( \kappa/\kappa_0 \), when \( \kappa_0 \) is a function of time. This assumption is equivalent to our former assumption that \( f(r) \) is a function of \( r/L \); i.e., we assume that \( \mathcal{F} \) and \( f \) preserve their shapes during the decay. Evidently \( \kappa_0 \sim 1/L \). Then the function can be written in the form

\[
\mathcal{F}(\kappa) = \frac{\overline{u^2}}{\kappa_0} \Phi \left( \frac{\kappa}{\kappa_0} \right).
\]

Then with \( \kappa/\kappa_0 = \xi \) and

\[
\frac{\partial \mathcal{F}}{\partial t} = \frac{\Phi}{\kappa_0} \frac{d\overline{u^2}}{dt} - \frac{\overline{u^2}}{\kappa_0^2} \Phi \frac{d\kappa_0}{dt} - \frac{\overline{u^2}}{\kappa_0^2} \Phi' (\xi) \frac{d\kappa_0}{dt},
\]

equation (12) becomes

\[
\left( \frac{1}{\kappa_0} \frac{d\overline{u^2}}{dt} - \frac{\overline{u^2}}{\kappa_0^2} \frac{d\kappa_0}{dt} \right) \Phi - \frac{\overline{u^2}}{\kappa_0^2} \frac{d\kappa_0}{dt} \Phi' \xi + \mathcal{W}_x = 0,
\]  

(14)

where

\[
\mathcal{W}_x = -C [\overline{u^2}]^{3/2} \left[ \Phi \alpha \xi^\beta \int_0^\xi \Phi(\xi')^{3/2-\alpha} \xi'^{1/2-\beta} d\xi' \right. \\
- \Phi^{3/2-\alpha} \xi^{1/2-\beta} \int_\xi^{\infty} \Phi(\xi')^{3/2-\alpha} \xi'^{\beta} d\xi'.
\]
According to Loitsianskii's (1939) results, 
\[
\frac{1}{u^2} \frac{du^2}{dt} = \frac{5}{\kappa_0} \frac{d\kappa_0}{dt},
\]
and one obtains the equation 
\[
\xi^5 (\xi^{-4} \Phi)' = -5C \frac{[u^2]^{3/2}}{du^2} \left[ \Phi^{\alpha} \xi^2 I_0^\xi - \Phi^{3/2-\alpha} \xi^{1/2-\beta} I_\xi^\infty \right], \tag{15}
\]
where 
\[
I_0^\xi = \int_0^\xi \Phi(\xi')^{3/2-\alpha} \xi'^{1/2-\beta} d\xi' \quad ; \quad I_\xi^\infty = \int_\xi^\infty \Phi(\xi')^{\alpha} \xi^\beta d\xi'.
\]

Let us assume that \(4 \alpha + \beta < 5/2\) as, for example, in the case of Heisenberg (1946). Then for small values of \(\xi\), the right side of equation (15) is small in comparison with the term on the left side, and one has 
\[
\Phi (\xi) \equiv \text{const} \xi^4.
\]
If \(4 \alpha + \beta > 5/2\), \(\Phi\) begins with a lower power of \(\kappa\) than \(\kappa^4\), one can show that the integral \(\int_0^\infty r' f(r) dr\) does not converge, so that Loitsianskii's result is incorrect. I should like to investigate this second case in a later work. Let us assume, for the time being, that Loitsianskii's result is correct; therefore the first case prevails. Then it follows that \(\Phi\) or \(\Phi\) behaves as \((\kappa/\kappa_0)^4\) for small values of \(\kappa\) and is proportional to \((\kappa/\kappa_0)^{-5/3}\) for large values of \(\kappa\). For any definite choice of \(\alpha\) and \(\beta\), the differential equation (15) can be solved numerically. The result that \(\Phi \equiv \kappa^4\) for small values of \(\kappa\) was also found in a different way by C. C. Lin (personal communication).

For the time being I propose an interpolation formula as follows: 
\[
\Phi (\xi) = \text{const}. \frac{\xi^4}{(1 + \xi^2)^{17/8}}. \tag{16}
\]
This interpolation formula represents correctly \(\Phi (\xi)\) for small and large values of \(\xi\) and has the advantage that all calculations can be carried out analytically by use of known functions. The results are as follows:
The $K$'s are Bessel functions with imaginary argument. For small values of $K_0 r$, 

$$f(K_0 r) = 1 - \frac{\Gamma(2/3)}{\Gamma(4/3)} \left( \frac{K_0 r}{2} \right)^{2/3},$$

as suggested by Kolmogorov's theory.

I have compared these results with the measurements of Liepmann, Laufer and Liepmann (1948) carried out at the California Institute of Technology with the financial assistance of the National Advisory Committee for Aeronautics. These observations were made in the 10-foot wind tunnel of the Guggenheim Aeronautical Laboratory, using a grid whose mesh size was $M = 4''$. The measurements were made at a distance $x = 40.4 M$ from the grid. Fig. 2 shows the comparison of calculated and measured values for the spectral function $\mathfrak{F}_1(k)$. It has to be taken into account that the observed values of $\mathfrak{F}_1(k)$ have large scatter; the deviation for high values of $k_1$ corresponds to the beginning influence of viscosity. Fig. 3 gives a comparison between measured and calculated values of the correlation function $g(r)$. This function is chosen because the observations are more accurate than in any other case. It is seen that the agreement is almost too good in view of the assumptions made above. One must remark that there is only one arbitrary constant in the formula for $g$, viz., the constant $K_0$ which determines the scale of the turbulence. It is true that some of the data of Liepmann, et al. (1948) do not show such a good agreement. The agreement is excellent for values of $g$ larger than 0.1, but after that the measured values are higher than the calculated ones. Possibly some oscillations existing in the wind tunnel stream were interpreted as turbulence, or the turbulence is not quite isotropic.

The N. A. C. A. has kindly granted permission for these data to be used here before appearing in its official publications.
I believe that the merits of my deduction are:

a) The assumptions involved are exactly formulated;
b) The specific assumptions of Heisenberg's theory concerning the transition function are not used;
c) The actual process of decay is considered;
d) The analysis is extended to the lower end of the turbulence spectrum.

Concerning the case of large values of $\kappa$ (small values of $r$), Kovász-nay (1948) introduced an interesting assumption which is more...
restricting than my fourth assumption. Obviously \( \int_0^\infty W_\kappa d\kappa \) is the total energy transferred by the Reynolds stresses from the interval \( 0 \rightarrow \kappa \) to the interval \( \kappa \rightarrow \infty \). Kovásznay assumes—following Kolmogoroff’s (1941) arguments—that this quantity is a function of \( \Phi(\kappa) \) and \( \kappa \) only. Then for dimensional reasons

\[
\int_0^\infty W_\kappa d\kappa = \text{const} \ \Phi^{3/2} \kappa^{5/2}.
\]

This assumption appears to be correct for large values of \( \kappa \). However, when the assumption is extended to the range of small values of \( \kappa \), and one substitutes \( W_\kappa \) in equation (8), one can calculate easily \( \Phi(\kappa) \). Neglecting the viscous term, one obtains the relation

\[
\Phi(\kappa/\kappa_0) = \text{const.} \ \frac{(\kappa/\kappa_0)^4}{[1 + \Phi^{1/2} (\kappa/\kappa_0)^{3/2}]^{17/2}}.
\]
The right side of equation (19) behaves as my corresponding equation (17) for small and large values of $k/k_0$. It will be interesting to see how far different transition from small to large values influences the accordance with observation.

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