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INTERNAL WAVES IN THE GULF OF CALIFORNIA*

By

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Between February 13 and March 19, 1939 the “E. W. Scripps,” research vessel of the Scripps Institution of Oceanography occupied fifty-three oceanographic stations in the Gulf of California. On the basis of observations of temperature and salinity which were taken at every station at a series of depths a chart of the dynamic topography of the surface relative to the 1500-decibar surface was derived, but this chart does not indicate any regular exchange of water between the Gulf and the adjacent part of the Pacific. It shows a sequence of separate highs and lows which are associated with alternating downward and upward displacements of the isopycnic surfaces and suggests the existence of internal waves with amplitudes as one-tenth the depth of the Gulf.

Let \( \rho \) be the density, depending upon the temperature and salinity of the water. The vertical displacements, as represented by \( \sigma_t \) curves \( (\sigma_t = 10^3(\rho - 1)) \) are clearly seen in a longitudinal section (Figure 7) based on data from stations along the center line of the Gulf, as shown on the small map to the left in the figures. It should be observed that the vertical scale is greatly exaggerated relative to the horizontal, the ratio between vertical and horizontal scales being equal to 1/800. The profiles of a series of isobaric surfaces relative to the 1500-decibar surface are shown to the right in the figure.

These observations have been discussed by H. U. Sverdrup (1940; in press) who concludes that the wavy character of the isobaric surfaces might be due to the presence of a standing internal wave, the period of which would lie between 6.3 days and 7.65 days, probably closer to the former value. This explanation is consistent with boundary conditions. The wave would be of first (fundamental)

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order with respect to the vertical axis, the vertical displacement vanishing at surface and bottom only, and three nodes inside the Gulf make it of fourth order with respect to the horizontal axis. A node at the mouth of the Gulf indicates that one has to deal with a free oscillation. The wave length of 1000 kilometers near the mouth of the Gulf indicates that the two progressive waves which combine to the standing wave there have velocities of propagation of 175 cm/sec which decrease towards the closed end of the Gulf. For further description of these waves and internal waves generally refer-

![Figure 7. Vertical distribution of density (\(\sigma_t\)) and profiles of isobaric surfaces referred to the 1500-decibar surface along a section in the Gulf of California, as shown on the map at the left in the figure. (After H. U. Sverdrup.)](image)

ence should be made also to Defant (1929), Sverdrup, Johnson and Fleming (in press), and W. H. Munk (in press).

Internal waves are accompanied by large horizontal velocity components at the bottom. It will be shown later (Figure 9) that the horizontal velocity at the bottom of the Gulf attains a maximum value of 20 cm/sec which is sufficiently large to exert a pronounced influence upon the processes of sedimentation. Within a standing wave the horizontal velocity component reaches a maximum near the location of nodes and one may expect that only large-sized particles settle in their vicinity. On the other hand, anti-nodes are characterized by very small horizontal currents which would permit small particles to be deposited at the bottom.

Such an influence of internal waves upon the distribution of sediments was pointed out by R. R. Revelle (1939) who found the aver-
age median diameters of bottom samples in the Gulf to vary in a regular manner along a north-south direction, corresponding to an internal wave with three nodes. This evidence is not conclusive, but it suggests that the particular wave-form of fourth order with respect to the horizontal axis does not constitute an isolated phenomenon but appears to be of common occurrence. The evidence would be greatly strengthened if it could be shown by means of theoretical considerations that the observed oscillation represents one of the possible natural oscillations in the Gulf.

Such a wave might be caused by an outside periodic disturbance the period of which would have to be closely related to one of the free periods of oscillation in the Gulf. The lunar-fortnightly tide has a period of 13.6 days and a theoretical amplitude which is one sixth the amplitude of the principle semi-diurnal lunar tide. If the length of one of the natural periods of the Gulf would correspond closely to 13.6 days an oscillation with this period would be dominant, but if a natural period close to 13.6 days does not exist, the lunar-fortnightly tide may cause an oscillation of approximately half this length, 6.8 days, corresponding to the observed period of oscillation of the Gulf.

Hence, it will be necessary to investigate theoretically the question whether the picture which has been derived from observations is in agreement with theoretical considerations and whether lunar forces may be responsible for the formation and maintenance of an internal wave of the observed type.

A body, defined by its distribution of mass and its dimensions, has a definite set of natural frequencies. These frequencies can be easily calculated in the case of uniform bodies of simple geometrical shape but the corresponding calculation becomes very complicated when applied to the waters of the Gulf of California within which the distribution of density is a function of depth (Figure 9) and the boundaries of which are determined by the topographic features of the Gulf (Figure 8). J. E. Fjeldstad (1933) has developed the theory of progressive internal waves in a channel of constant depth, provided that the density distribution does not change in a horizontal direction. W. H. Munk (in press) has treated the cases of progressive or standing waves, assuming constant depth and appreciable horizontal density gradient and of standing waves, assuming variable depth but negligible horizontal density gradient. For rigorous derivations reference is made to these papers, and here it will be attempted only to outline the latter theory, to explain its application, and to discuss the results pertaining to the Gulf.

Previous investigations of wave phenomena (Defant, 1929) have
shown that in a long narrow channel the earth's rotation will tend to produce oscillations in a plane perpendicular to the length of the channel, but its effect upon the chief wave motion along the channel will be small and can in first approximation be neglected. For that reason the problem reduces to a two-dimensional analysis.

We shall assume the x-axis along the direction of propagation of the wave, the z-axis perpendicular to the surface, where \( z = 0 \) represents the undisturbed surface, \( z = H \) the bottom. (Figure 8.) Let \( u \) and \( w \) represent the components of the velocity of the water along the \( x \) and \( z \) axes respectively, \( \rho \) the density, \( p \), the pressure, and \( \eta \) the vertical displacement. Let \( \rho_0(z) \) denote the density in equilibrium condition and define \( \rho_1 \) by

\[
\rho = \rho_0(z) + \rho_1(x, z, t)
\]

Let

\[
p_1 = g \int_0^z \rho_1(x, z, t) \, dz
\]

Neglecting second-order terms, the fundamental equations of hydrodynamics take the form

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho_0} \frac{\partial p_1}{\partial x} = 0 \tag{1a}
\]

\[
\frac{\partial w}{\partial t} + g \frac{\rho_1}{\rho_0} + \frac{1}{\rho_0} \frac{\partial p_1}{\partial z} = 0 \tag{1b}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1c}
\]

\[
\frac{\partial \rho_1}{\partial t} + w \frac{\partial \rho_0}{\partial z} = 0 \tag{1d}
\]

To investigate a progressive or standing wave with period \( T = \frac{2\pi}{\sigma} \), we assume:

\[
\begin{array}{ll}
\text{Progressive wave} & \text{Standing wave} \\
\begin{align*}
u (xzt) &= U (xz) \sin (kx - \sigma t), & u (xzt) &= U (xz) \cos (\sigma t) \\
w (xzt) &= W (xz) \cos (kx - \sigma t), & w (xzt) &= W (xz) \cos (\sigma t) \\
p_1 (xzt) &= P_1 (xz) \sin (-kx + \sigma t), & p_1 (xzt) &= P_1 (xz) \sin (\sigma t) \\
\rho_1 (xzt) &= R_1 (xz) \sin (-kx + \sigma t), & \rho_1 (xzt) &= R_1 (xz) \sin (\sigma t)
\end{align*}
\end{array}
\tag{2}
\]
Substitution of either set of assumptions into equation (1) eliminates the function of time, and the remaining four equations in four unknowns can be solved for $W(xz)$:

$$\frac{\partial^2 W(xz)}{\partial z^2} - g \frac{\varphi(z)}{\sigma^2} \frac{\partial^2 W(xz)}{\partial x^2} = 0$$  \hspace{1cm} (3)$$

where

$$\varphi = -\frac{1}{\rho_0} \frac{\partial \varphi_0}{\partial z}$$  \hspace{1cm} (3b)$$

The substitution $W(xz) = W(z) \sin (kx)$ reduces the partial differential equation (3) to the total differential equation:

$$\frac{d^2 W(z)}{dz^2} + \left(\frac{1}{c}\right)^2 g \varphi W(z) = 0$$  \hspace{1cm} (4)$$

where the wave length $\lambda = \frac{2\pi}{k}$ and the velocity of propagation, $c = \frac{\lambda}{T}$.

This is the differential equation derived by Fjeldstad for the case of constant depth. It can be shown, however (Munk, in press), that this substitution $W(xz) = f(z) \cdot F(x)$ is inconsistent with the general problem and consistent only for a basin of constant depth, wherefore, assumptions (2) leading to equation (3) must be used when treating the case of variable depth.

Assumptions (2) referring both to standing and progressive waves, satisfy the basic laws of hydrodynamics and lead to differential equation (3). This is true because the standing wave may be considered a superposition of two progressive waves of equal amplitude but opposite direction, and both progressive waves are proper solutions, hence their sum, the standing wave, must also be a solution. On the other hand, the two wave-forms behave in a very different manner with regard to boundary conditions and it can be shown (Munk, in press) that the solution referring to progressive waves cannot be applied to a basin of variable depth. This indicates that a standing wave can exist in a basin in which neither of the two progressive waves can exist, which may be considered as making up the standing wave.

In order to obtain values for the periods of natural frequency, $\sigma$ must be computed from the differential equation (3), subject to boundary conditions:

$$W = 0 \text{ at } x = 0$$  \hspace{1cm} (5)$$

since a node is located at the mouth of the Gulf (Figure 7), and
Since the vertical motion caused by an internal wave at the surface is always small enough to be neglected. Finally, the kinematic boundary condition at the bottom, namely, that all flow must take place along the boundary, is expressed by

\[ W = mU \text{ along the bottom, where } m \text{ denotes the slope.} \] (7)

Let \( W \) be expanded in a Fourier series:

\[ W(xz) = \sum_{i=1}^{n} W_i(z) \sin(i \pi x/L) \hat{C}_i(o) \] (8)

where \( k_i = i \pi / 2L \), \( L \), the length of the basin, and \( C_i(o) \) a constant so chosen that \( \hat{W}_i(o) = 1 \left( \hat{W}_i(z) = \frac{dW_i}{dz} \right) \)

With these substitutions we obtain for equation (3)

\[ \frac{d^2W_i(z)}{dz^2} + g \varphi(z) k_i^2 W_i(z) = 0 \quad (i = 1, 2, \ldots n) \] (9)

Boundary condition (5) is already satisfied. Boundary condition (6) gives

\[ W_i(o) = 0 \] (10)

Boundary condition (7), with the aid of the equation of continuity, yields

\[ \sum_{i=1}^{n} \left[ W_i(z) \sin(k_i x) + \hat{W}_i(z) \frac{m(x) \cos(k_i x)}{k_i} \right] \hat{C}_i(o) = 0 \] (11)

In the Fourier summation (8), \( n \) must be at least equal to two in order to satisfy all boundary conditions of a basin of variable depth and must be larger than two to give a good approximation. By making \( n \) sufficiently large we can approximate as closely as we wish any velocity distribution corresponding to an observed wave.

We note that equation (9) represents \( n \) equations of the same type as equation (4), which Fjeldstad has developed for the case of constant depth. Each of these equations describes on internal wave of wave length \( \lambda_i = \frac{4L}{k_i} \) and can be integrated numerically according to the method used by Fjeldstad (1933). In this manner we obtain values for \( W_i(z) \) and \( \hat{W}_i(z) \) at all depths. The internal wave can then be considered as the result of \( n \) internal waves of equal period but different wave lengths, superimposed upon one another.
In order to apply boundary condition (11) we shall divide the length \( L \) of the basin into \( r \) equidistant intervals with values of \( x \) ranging from \( x_l = 0 \) to \( x_r = L \) (Figure 8). By means of an accurate topographic section of the basin one can find for all values of \( x_j \) (\( j = 1, 2, 3 \ldots r \)) the corresponding slope \( m_j \) and depth \( z_j \). Let us then assume a definite value for the parameter \( \sigma \) and obtain \( W_i(z) \) and \( \dot{W}_i(z) \), for all values of \( z_j \) by means of numerical integrations. If we define

\[
K_{ji} = W_i(z_j) \sin (k_i x_j) + \dot{W}_i(z_j) \frac{m(z_j) \cos (k_i x_j)}{k_i}
\]

boundary condition (11) becomes simply

\[
\sum_{j=1}^{r} \sum_{i=1}^{n} K_{ji} \dot{C}_i (\sigma) = 0
\]

Munk (in press) has devised a method to test how well boundary condition (13) is satisfied. It calls for the evaluation of a determinant \( D \) which will vanish for every value of \( \sigma \) which corresponds to a natural frequency. This, however, is true only for an infinite value of \( n \), for otherwise a Fourier series will give an approximation only to the actual velocity distribution, and \( D \) will not vanish completely but will reach a minimum.
For \( n = 3 \),
\[
D = \begin{vmatrix}
\sum_{j=1}^{r} K_{j1}K_{j1} & \sum_{j=1}^{r} K_{j1}K_{j2} & \sum_{j=1}^{r} K_{j1}K_{j3} \\
\sum_{j=1}^{r} K_{j2}K_{j1} & \sum_{j=1}^{r} K_{j2}K_{j2} & \sum_{j=1}^{r} K_{j2}K_{j3} \\
\sum_{j=1}^{r} K_{j3}K_{j1} & \sum_{j=1}^{r} K_{j3}K_{j2} & \sum_{j=1}^{r} K_{j3}K_{j3}
\end{vmatrix}
\]  

\[
(14)
\]

\( D \) can now be calculated for various values of \( \sigma \) and the natural frequencies can be determined because the value of \( D \) must be at a minimum when \( \sigma \) corresponds to a natural frequency.

The bottom topography of the Gulf of California is quite complicated but observations at the deepest stations indicate that to a considerable distance from the mouth continuous communication between the various depressions is present above 2000 meters. It seemed therefore justifiable to choose an “effective” depth which remains close to 1800 meters throughout the larger portion of the Gulf, but decreases regularly in the inner part.

At about 29° latitude the channel becomes very shallow and two islands, San Esteban and San Lorenzo, rise above the surface. Waves will be reflected before they reach this point or lose their energy by friction and interference. Hence we may consider this the inner end of the Gulf, and may neglect the portion north of the islands, although the Gulf deepens and widens again, and extends 350 kilometers further inland.

It is generally known that the “effective” length of a bay with regard to a wave phenomenon is not identical with its actual length, but depends upon its width at the open end. A correction formula has been derived which, in this case, gives an “effective” length of 1000 kilometers, roughly 20 per cent longer than the actual length.

Fjeldstad’s theory for basins of constant depth was applied to the outer portion of the Gulf of California. Figure 9 illustrates the results obtained by this theory. They compare favorably with those obtained from observations.

<table>
<thead>
<tr>
<th>Theory</th>
<th>c</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>167 cm/sec</td>
<td>675 m</td>
</tr>
<tr>
<td>Observations</td>
<td>175 cm/sec</td>
<td>500–600 m</td>
</tr>
</tbody>
</table>

where \( c \) represents the velocity of propagation, \( H \) the depth of maximum amplitude of oscillation.
The calculation of $D$ was carried out for three values of $i$ only. For investigating an approximately weekly period values of $i = 4, 5,$ and $7$, corresponding to wave-lengths near the mouth of 1000 km, 800 km, and 570 km were chosen, whereas in the case of the fortnightly period values of $i = 3, 4,$ and $5$ were chosen corresponding to wave-lengths of 1333 km, 1000 km, and 800 km. These particular components were most satisfactory because they gave a low value for $D$, as illustrated in the following tables:

<table>
<thead>
<tr>
<th>$T = 6.4$ days</th>
<th>5-6-7</th>
<th>3-4-5</th>
<th>4-5-6</th>
<th>4-5-7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D \times 10^{-27}$</td>
<td>130</td>
<td>123</td>
<td>63</td>
<td>25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$T = 14$ days</th>
<th>2-3-4</th>
<th>2-3-5</th>
<th>2-4-5</th>
<th>3-4-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D \times 10^{-27}$</td>
<td>26</td>
<td>54</td>
<td>6.4</td>
<td>4.0</td>
</tr>
</tbody>
</table>

A further examination (Munk, in press) shows that in both cases a wave dominates which near the mouth has a wave length of 1000 km.
km, decreasing toward the closed end. The nodal regions of both waves must approximately coincide and the distribution of sediments may be consistent with the presence of either wave.

In a vertical direction the weekly wave is primarily of the first order, whereas the fortnightly wave is of the second order. The observed distribution of density indicates that a first-order wave dominates but the presence of a second-order wave is not excluded and may account for the fact that the observations appear to give a smaller value of $H$, the depth of maximum vertical oscillation.

The main results of the numerical integrations are presented in Figure 10 according to which minima correspond to periods of 7 days and 14.6 days, respectively, and their relations to the lunar fortnightly tidal period.

![Figure 10](image-url)

Figure 10. Results obtained by numerical integration indicating two periods in the Gulf of California of natural frequencies of 7 days and 14.6 days, respectively, and their relations to the lunar fortnightly tidal period.
the case of the 14.8 days, and that the seven-day period, therefore, may dominate. Thus, the theoretical examination fully confirms Sverdrup's interpretation of the observations from the Gulf of California.

The writer wishes to acknowledge the help of Dr. H. U. Sverdrup, who suggested this investigation and without whose advice and encouragement it could not have been completed. The writer is also indebted to Dr. H. Bateman for his help in finding a solution to the general differential equation, and to Dr. L. Lek and Mr. F. Brunner for having carried out many of the tedious calculations.

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